



Abstract: We prove a conjecture of Alon on the number of independent sets in a regular graph, and then generalize to graph homomorphisms.

Independent Sets

Let $G = (V, E)$ be a graph. An **independent set** is a subset of the vertices with no two adjacent.

Question. In the family of N -vertex, d -regular graphs G , when is the number of independent sets maximized?

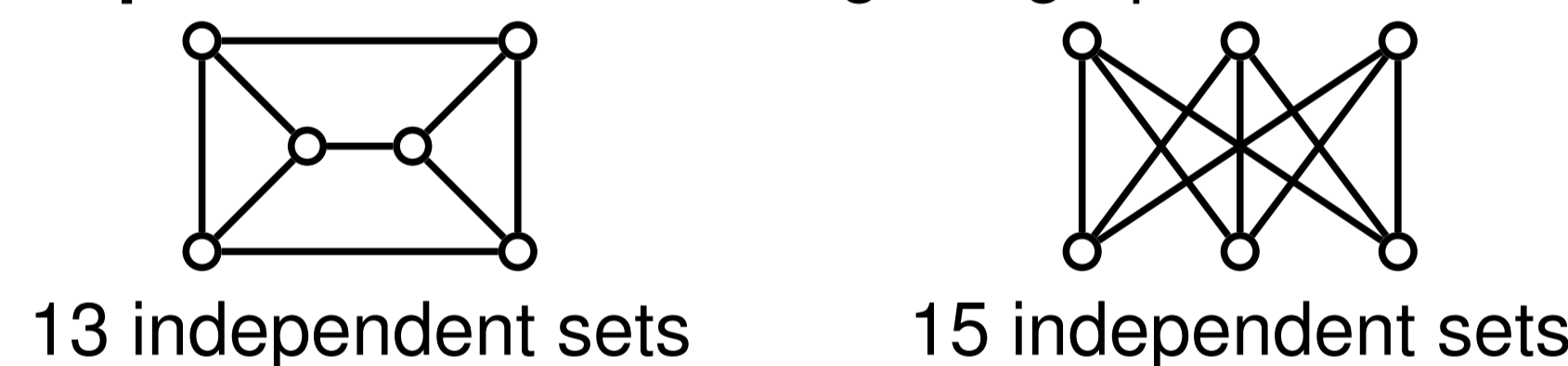
Theorem 1 (Zhao [4]; conjectured by Alon [1] in 1991 and proved for G bipartite by Kahn [3] in 2001). For any N -vertex, d -regular graph G ,

$$i(G) \leq i(K_{d,d})^{N/2d} = (2^{d+1} - 1)^{N/(2d)},$$

where $i(G)$ denotes the number of independent sets of G . Note equality holds if G is a disjoint union of $K_{d,d}$'s.

All results have weighted generalizations.

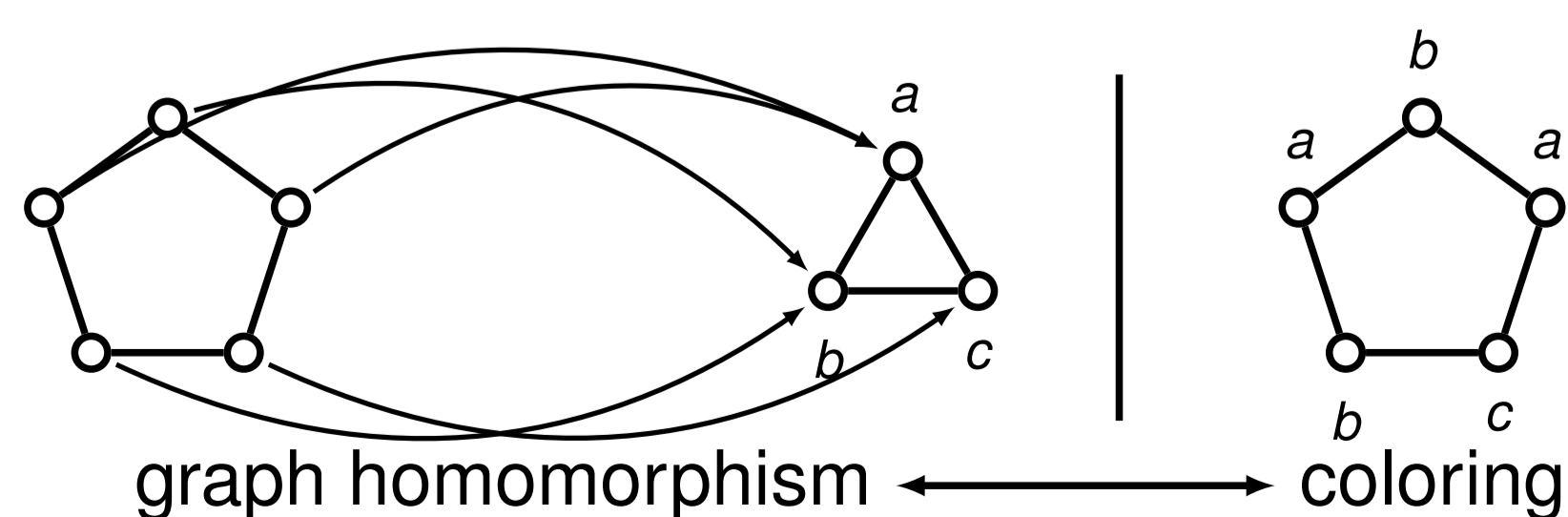
Example: Two 6-vertex 3-regular graphs:



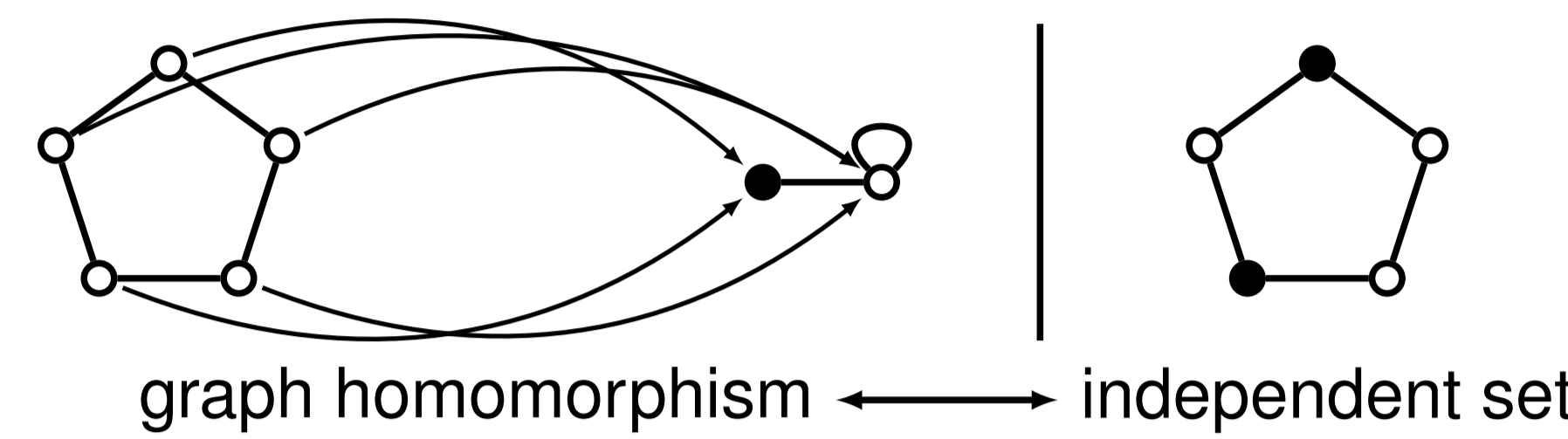
Graph Homomorphisms

For graphs G and H (allowing loops for H), a **graph homomorphism** from G to H is a map from $V(G)$ to $V(H)$ so that all every edge of G gets carried to some edge of H . Denote the set of graph homomorphisms from G to H by $\text{Hom}(G, H)$.

Graph homomorphisms generalize graph colorings by choosing $H = K_k$ for a k -coloring.



Graph homomorphisms also generalize the independent sets, by choosing $H = \bullet \rightarrow \bullet$.



Goal. Generalize Thm 1 to graph homomorphisms.

Question. For a fixed graph H (allowing loops), in the family of N -vertex, d -regular simple graphs G , when is $|\text{Hom}(G, H)|$ maximized?

It is known that Theorem 1 generalizes at least in the bipartite case.

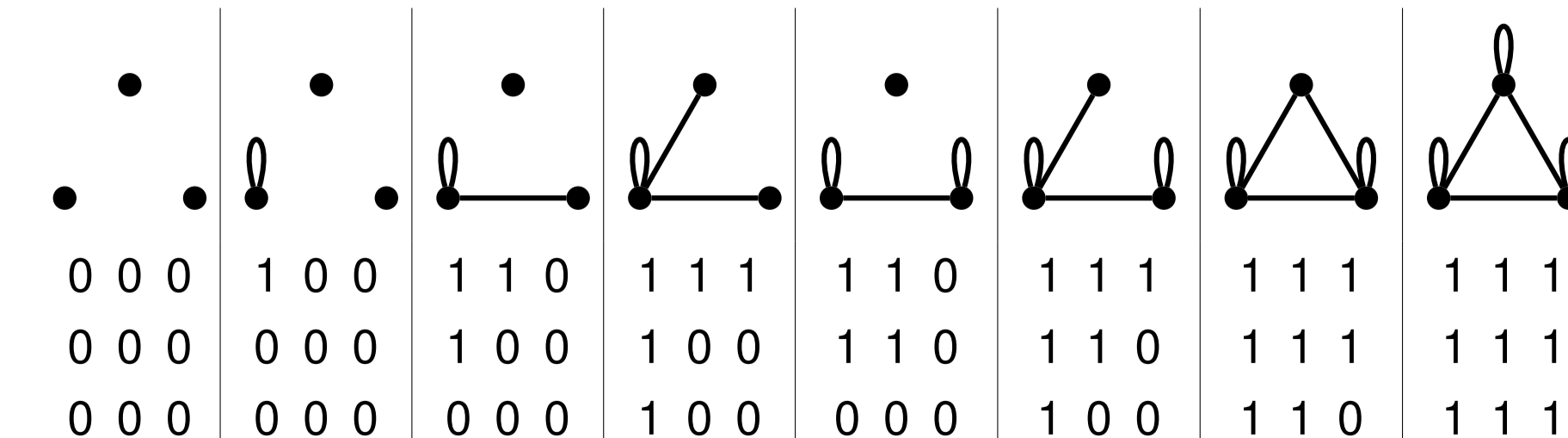
Theorem 2 (Galvin & Tetali [2]). For any H (allowing loops), and any N -vertex, d -regular bipartite graph G ,

$$|\text{Hom}(G, H)| \leq |\text{Hom}(K_{d,d}, H)|^{N/(2d)}.$$

Theorem 3 ([5]). The bipartite condition on G in Theorem 2 can be dropped if H is a *bipartite swapping target*.

Bipartite Swapping Targets

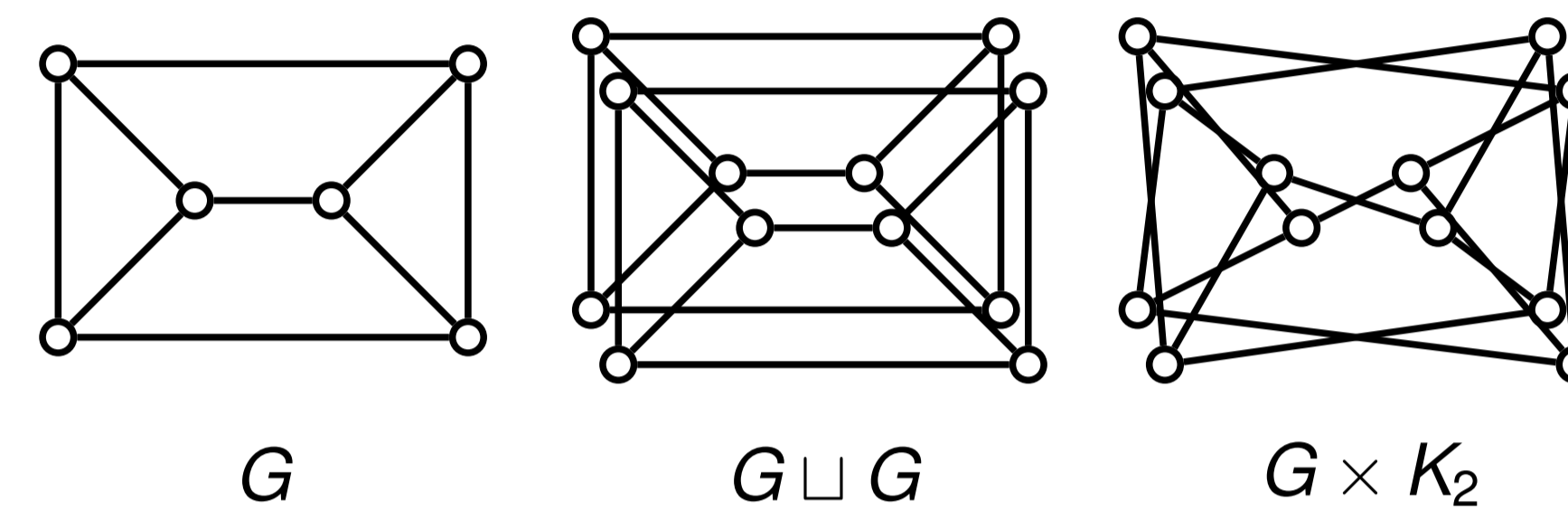
- The family of bipartite swapping targets is defined by some technical conditions that we omit here.
- H is a bipartite swapping target if and only if some auxiliary graph is bipartite, so the property is easy to check.
- **An easy-to-describe subclass:** Any graph whose adjacency matrix can be written so that the 1's form a Young diagram is a bipartite swapping target. E.g., all such 3-vertex graphs:



Bipartite Swapping Trick

Idea: Reduce general graph G to bipartite graph by comparing $G \sqcup G$ with $G \times K_2$ and finding an injection

$$\text{Hom}(G \sqcup G, H) \rightarrow \text{Hom}(G \times K_2, H)$$

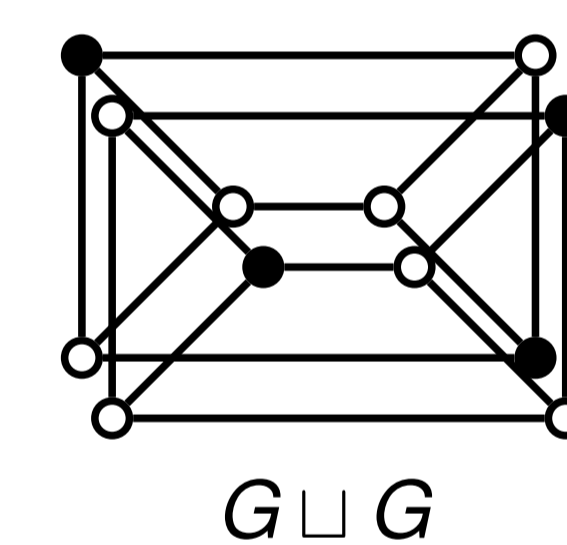


Suppose that we have such an injection, since $G \times K_2$ is bipartite, Theorem 2 would imply that

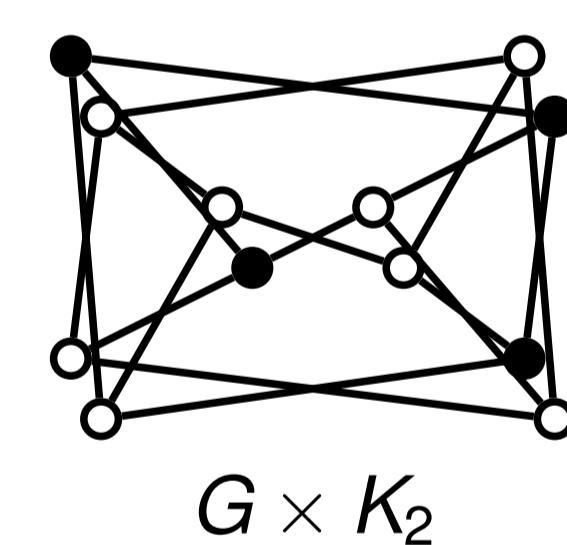
$$|\text{Hom}(G, H)| = |\text{Hom}(G \sqcup G, H)|^{1/2} \leq |\text{Hom}(G \times K_2, H)|^{1/2} \leq |\text{Hom}(K_{d,d}, H)|^{N/(2d)}.$$

Construction of injection: (for independent sets)

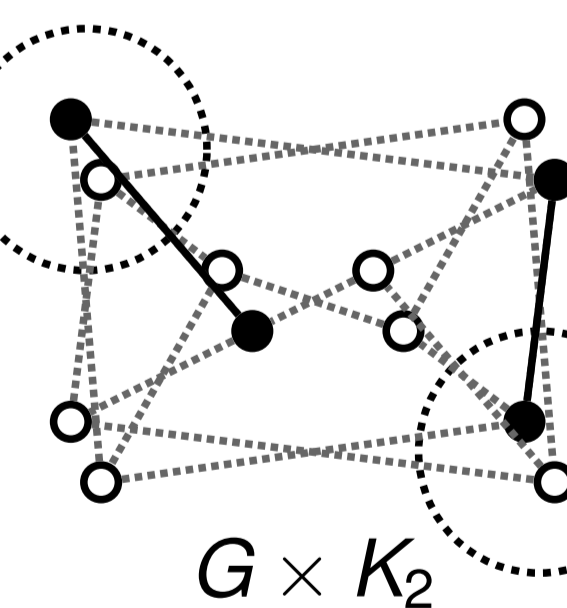
Start with an independent set (black vertices) of $G \sqcup G$



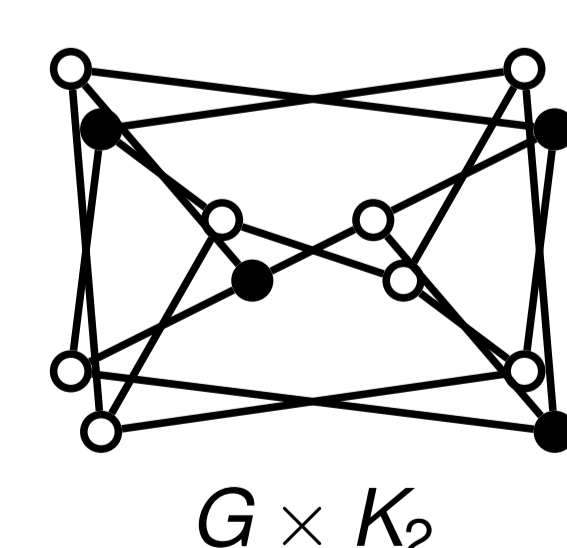
Cross edges to get $G \times K_2$. However, the same subset of vertices might no longer be an independent set.



The set of "violated edges" turns out to form a bipartite graph. Select the lexicographically first set of vertex-pairs that gives a bipartition of the violated edges.



Swap the vertices in the pairs selected in the previous step. This gives us an independent set of $G \times K_2$.



Remark: Bipartite swapping property of H allows this technique to be generalized to $\text{Hom}(G, H)$.

Stable Set Polytope

The **stable set polytope** of G is defined as the convex hull in $\mathbb{R}^{V(G)}$ of the characteristic vectors of the independent sets of G .

Theorem 4 ([5]). For any N -vertex, d -regular graph G , we have

$$i_V(G) \leq i_V(K_{d,d})^{N/(2d)} = \left(\frac{2d}{d}\right)^{-N/(2d)},$$

where $i_V(\cdot)$ is the volume of the stable set polytope.

Graph Colorings

Unfortunately, K_d does not pass our test for a bipartite swapping target, so our technique cannot solve the graph coloring case. However, using the bipartite swapping trick, we can still prove the following result.

Theorem 5 ([5]). For every N -vertex, d -regular graph G ,

$$|\text{Hom}(G, K_q)| \leq |\text{Hom}(K_{d,d}, K_q)|^{N/(2d)} \quad (1)$$

for all sufficiently large q . Note that $|\text{Hom}(\cdot, K_q)|$ counts the number of proper q -colorings.

Conjecture 6. The inequality (1) holds for all q .

Acknowledgments: This research was done at the Duluth REU and the MIT UROP. I thank Joseph Galiliani and Michel Goemans for their mentorship.

References

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