The Green–Tao theorem and a relative Szemerédi theorem

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Based on joint work with David Conlon (Oxford) and Jacob Fox (MIT)

Green–Tao Theorem (arXiv 2004; Annals of Math 2008)

The primes contain arbitrarily long arithmetic progressions (AP).

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Szemerédi's Theorem (1975)

Every subset of $\mathbb{N}$ with positive density contains arbitrarily long APs.

(upper) density of $A \subset \mathbb{N}$ is $\limsup_{N\to\infty} \frac{|A \cap [N]|}{N}$

$[N] := \{1, 2, \ldots, N\}$

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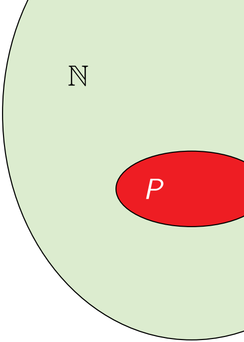
$P =$ prime numbers

Prime number theorem: $\frac{|P \cap [N]|}{N} \sim \frac{1}{\log N}$

Proof strategy of Green–Tao theorem



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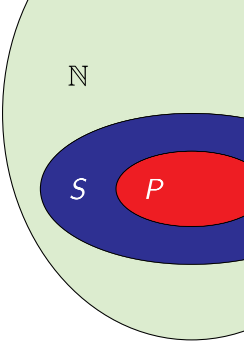

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$P \subseteq S$ with positive relative density, i.e., $\dfrac{|P \cap [N]|}{|S \cap [N]|} > \delta$

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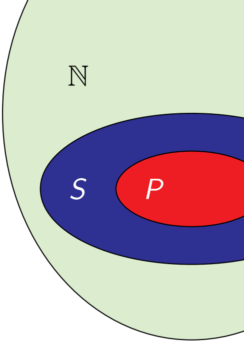

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Step 1:

Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of S with positive relative density contains long APs.

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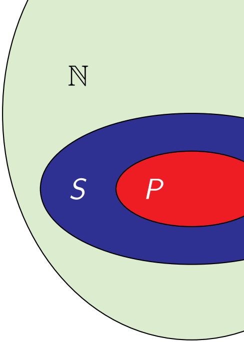

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Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of S with positive relative density contains long APs.

Step 2: Construct a superset of primes that satisfies the pseudorandomness conditions. (Goldston–Yıldırım sieve)

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What pseudorandomness conditions?

Green–Tao:

1. Linear forms condition
2. Correlation condition

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A natural question (asked by Gowers & Green)

Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

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Green–Tao:

1. Linear forms condition
2. Correlation condition $\leftarrow$ no longer needed

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Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

Theorem (Conlon, Fox, Z.)

Yes! A weaker linear forms condition suffices.

Szemerédi's theorem

Host set: $\mathbb{N}$

Relative Szemerédi theorem

Host set: some sparse subset of integers

Conclusion: relatively dense subsets contain long APs

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Random host set

- Kohayakawa–Łuczak–Rödl '96 — *3-AP*, $p \gg N^{-1/2}$
- Conlon–Gowers '10+
Schacht '10+ — *k-AP*, $p \gg N^{-1/(k-1)}$

Pseudorandom host set

- Green–Tao '08 — *linear forms + correlation*
- Conlon–Fox–Z. '13+ — *linear forms*

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Roth's theorem

Roth's theorem (1952)

If $A \subseteq [N]$ is 3-AP-free, then $|A| = o(N)$.

$[N] := \{1, 2, \ldots, N\}$

3-AP = 3-term arithmetic progression

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Roth's original proof uses Fourier analysis.
Let us recall a graph theoretic proof.

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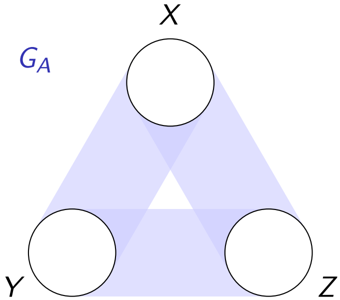

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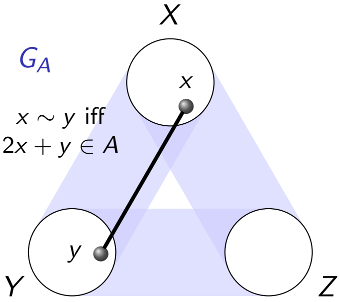

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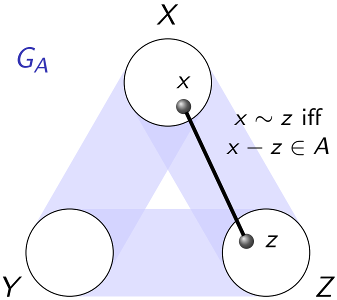

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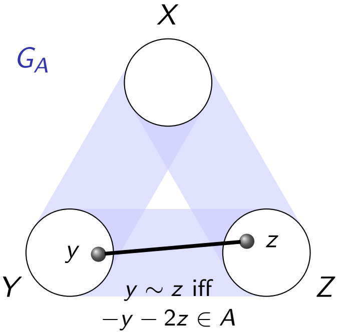

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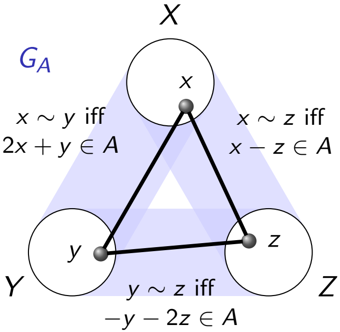

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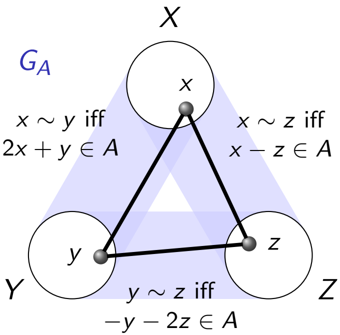
$$2x + y,\ x - z,\ -y - 2z \ \in A$$



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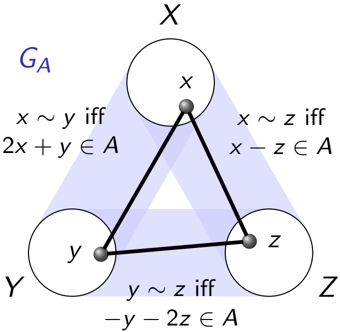

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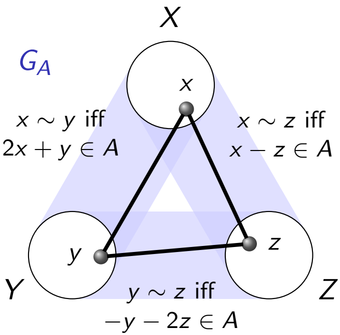

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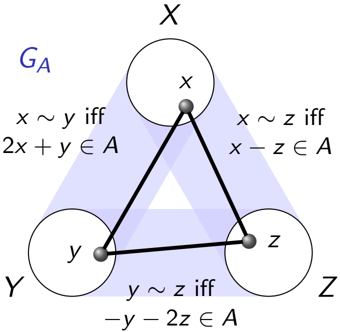

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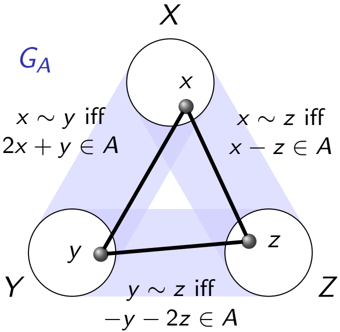

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Every edge of the graph is contained in exactly one triangle (the one with $x + y + z = 0$).

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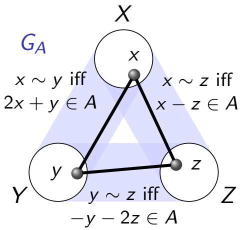
If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Constructed a graph with

- $3N$ vertices
- $3N|A|$ edges
- every edge lies in exactly one triangle



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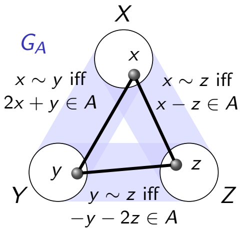

Theorem (Ruzsa & Szemerédi '76)

If every edge in a graph $G = (V, E)$ is contained in exactly one triangle, then $|E| = o(|V|^2)$.

(a consequence of the *triangle removal lemma*)

So $3N|A| = o(N^2)$. Thus $|A| = o(N)$.

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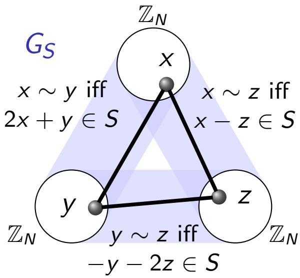

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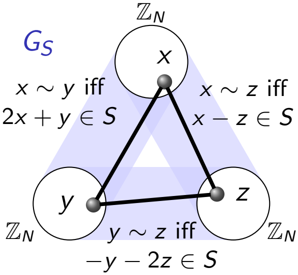
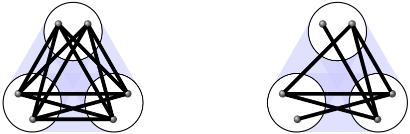
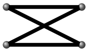


3-linear forms condition:
G_S has asymp. the same H-density as a random graph for every $H \subseteq K_{2,2,2}$



Analogy with quasirandom graphs

Chung-Graham-Wilson '89 showed that in constant edge-density graphs, many quasirandomness conditions are equivalent, one of which is having the correct C_4 count



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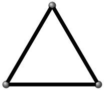
In sparse graphs, the Chung–Graham–Wilson equivalences do **not** hold.

Our results can be viewed as saying that:

Many extremal and Ramsey results about H (e.g., $H = K_3$) in sparse graphs hold if there is a host graph that behaves pseudorandomly with respect to counts of the 2-blow-up of H.



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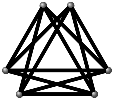

Relative Szemerédi theorem (Conlon, Fox, Z.)

Fix $k \geq 3$. If $S \subseteq \mathbb{Z}_N$ satisfies the k-linear forms condition, and $A \subseteq S$ is k-AP-free, then $|A| = o(|S|)$.

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$k = 4$: build a 4-partite 3-uniform hypergraph

Vertex sets $W = X = Y = Z = \mathbb{Z}_N$

- $wxy \in E \iff 3w + 2x + y \in S$
- $wxz \in E \iff 2w + x - z \in S$
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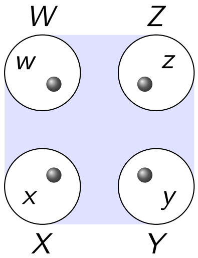

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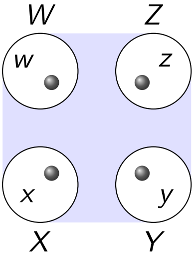

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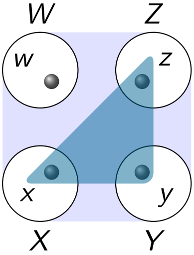

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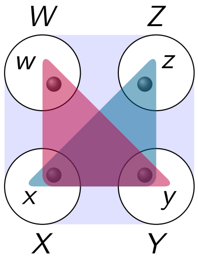

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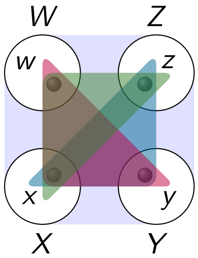

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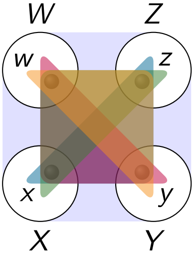

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By an averaging argument (Varnavides), we get many 3-APs:

Roth's theorem (counting version)

$\forall \delta > 0 \; \exists c > 0$ so that for sufficiently large N,
every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains at least cN^2 many 3-APs.

Transference

Start with

$$\text{(sparse)} \qquad A \subset S \subset \mathbb{Z}_N, \qquad\qquad |A| \geq \delta\,|S|$$

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$$\left(\frac{N}{|S|}\right)^3 |\{\text{3-APs in } A\}| \approx |\{\text{3-APs in } \widetilde{A}\}|$$

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$$\begin{aligned}\left(\frac{N}{|S|}\right)^3 |\{\text{3-APs in } A\}| &\approx |\{\text{3-APs in } \widetilde{A}\}| \\ &\geq cN^2 \qquad \text{[By Roth's Theorem]} \\ &\qquad\qquad\quad \text{(blackbox application)}\end{aligned}$$

$\Longrightarrow$ relative Roth theorem (also works for k-term AP)

Converting to functional language

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Sparse setting

Sparse set $A \subseteq S \subset \mathbb{Z}_N$ correspond to (normalized) indicator functions

$$\nu = \frac{N}{|S|} 1_S \qquad \text{and} \qquad f = \frac{N}{|S|} 1_A.$$

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More generally, we consider any (say that f is majorized by ν)

$$f \leq \nu \colon \mathbb{Z}_N \to [0, \infty) \quad \text{(pointwise inequality)}$$

with

$$\mathbb{E}\nu = 1 \qquad \text{and} \qquad \mathbb{E} f \geq \delta.$$

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Relative Roth theorem (Conlon, Fox, Z.)

$\forall \delta > 0\ \exists c > 0$ so that for sufficiently large N, if

- $\nu : \mathbb{Z}_N \to [0, \infty)$ satisfies the 3-linear forms condition, and
- $f : \mathbb{Z}_N \to [0, \infty)$ majorized by ν and $\mathbb{E} f \ge \delta$, then

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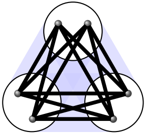
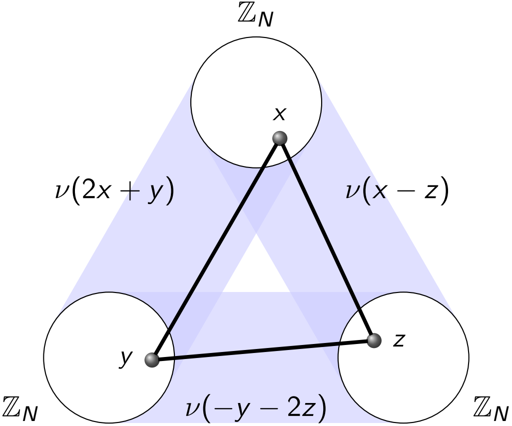
Recall $AP_3(f) = \mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x+d)f(x+2d)]$

Remark. The dependence of c on δ is the same.

3-linear forms condition

The density of $K_{2,2,2}$  in



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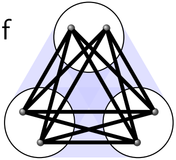

$$\begin{aligned}&\mathbb{E}[\nu(2x+y)\nu(2x'+y)\nu(2x+y')\nu(2x'+y')\cdot \\ &\quad \nu(x-z)\nu(x'-z)\nu(x-z')\nu(x'-z')\cdot \\ &\quad \nu(-y-2z)\nu(-y'-2z)\nu(-y-2z')\nu(-y'-2z')] = 1 + o(1)\end{aligned}$$

as well as if any subset of the 12 factors were deleted.

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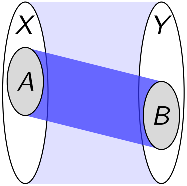
Using cut norm:

- Cheaper dense model theorem
- More difficult counting lemma

Cut norm for weighted bipartite graph (Frieze–Kannan):

$g \colon X \times Y \to \mathbb{R}$

$$\|g\|_{\square} := \frac{1}{|X|\,|Y|} \sup_{\substack{A \subseteq X \\ B \subseteq Y}} \left| \sum_{\substack{x \in A \\ y \in B}} g(x,y) \right|$$



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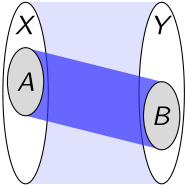

Cut norm for $\mathbb{Z}_N$: $f \colon \mathbb{Z}_N \to \mathbb{R}$

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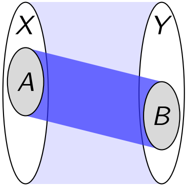

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Dense model theorem

Assume $\nu \colon \mathbb{Z}_N \to [0,\infty)$ satisfies $\|\nu - 1\|_\square = o(1)$.
Then $\forall\ 0 \le f \le \nu$, $\exists\ \widetilde{f} \colon \mathbb{Z}_N \to [0,1]$ s.t. $\|f - \widetilde{f}\|_\square = o(1)$.

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Proof of the general dense model theorem

1. Regularity-type energy-increment argument
   (Green–Tao, Tao–Ziegler)
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Specialized/simplified for the cut norm on $\mathbb{Z}_N$ (Z.)

Higher cut norms (for 4-term AP)

3-uniform weighted hypergraph $g\colon X \times Y \times Z \to \mathbb{R}$, define

$$\|g\|_{\square} := \frac{1}{|X|\,|Y|\,|Z|} \sup_{\substack{A \subseteq Y \times Z \\ B \subseteq X \times Z \\ C \subseteq X \times Y}} \left| \sum_{\substack{x \in X, y \in Y, z \in Z \\ (y,z) \in A \\ (x,z) \in B \\ (x,y) \in C}} g(x,y,z) \right|.$$

i.e., supremum taken over all 2-graphs between X, Y, Z

For $f\colon \mathbb{Z}_N \to \mathbb{R}$,

$$\|f\|_{\square,3} := \sup_{a,b,c\colon \mathbb{Z}_N \to [0,1]} \left| \mathbb{E}_{x,y,z \in \mathbb{Z}_N} f(x+y+z) a(y,z) b(x,z) c(x,y) \right|$$

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Weighted graphs $g, \widetilde{g} \colon (X \times Y) \cup (X \times Z) \cup (Y \times Z) \to \mathbb{R}$
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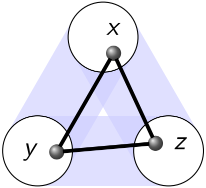
Triangle counting lemma (dense setting)

Assume $0 \le g, \widetilde{g} \le 1$. If $\|g - \widetilde{g}\|_\square \le \epsilon$, then

$$t(g) = t(\widetilde{g}) + O(\epsilon).$$



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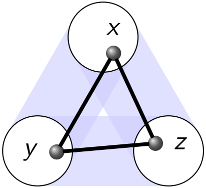

$$|\mathbb{E}[(g(x,y) - \widetilde{g}(x,y))1_A(x)1_B(y)]| \le \epsilon \quad \forall A \subseteq X, B \subseteq Y$$

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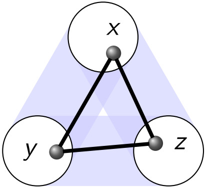

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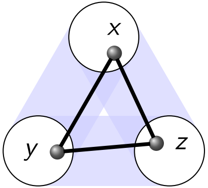

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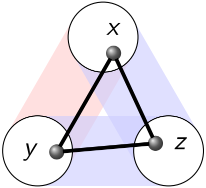

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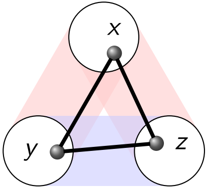

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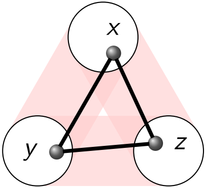

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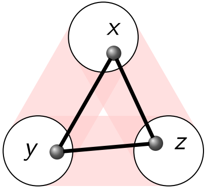

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This argument doesn't work in the sparse setting (g unbounded)

Sparse counting lemma

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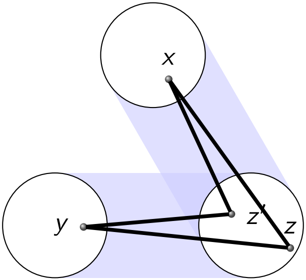
Proof ingredients

1. Cauchy-Schwarz
2. **Densification**
3. Apply cut norm/discrepancy (as in dense case)

Densification



$$\mathbb{E}[g(x,z)g(y,z)g(x,z')g(y,z')]$$

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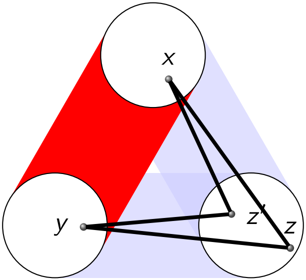

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Set $g'(x,y) := \mathbb{E}_{z'}[g(x,z')g(y,z')]$,
i.e., normalized codegrees

$g'(x,y) \lesssim 1$ for almost all (x,y)
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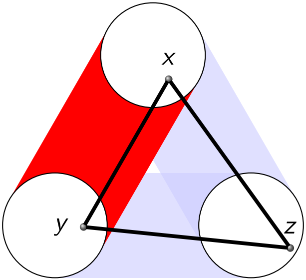

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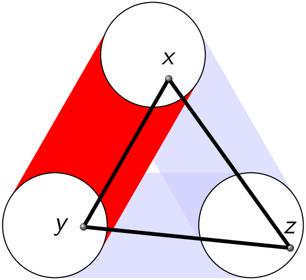

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Made $X \times Y$ dense. Now repeat for $X \times Z$ and $Y \times Z$.
Reduce to dense setting.

Transference

Start with $f \le \nu$

$$\text{(sparse)} \qquad f\colon \mathbb{Z}_N \to [0,\infty) \qquad \mathbb{E}f \ge \delta$$

Dense model theorem: one can approximate f (in cut norm) by

$$\text{(dense)} \qquad \widetilde{f}\colon \mathbb{Z}_N \to [0,1] \qquad \mathbb{E}\widetilde{f} = \mathbb{E}f$$

Counting lemma implies

$$AP_3(f) \approx AP_3(\widetilde{f}) \ge c \quad \text{[By Roth's Thm (weighted version)]}$$

$\Longrightarrow$ relative Roth theorem

Constructing the majorant for the primes

Step 1. Remove biases modulo small primes

Primes are biased on certain residue classes.
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W-trick: only consider primes in the residue class

$$1 \pmod W$$

where

$$W = \prod_{p \le w} p$$

for some very slowly growing w.

Constructing the majorant for the primes

Step 2. Goldston–Yıldırım sieve

von Mangoldt function

$$\Lambda(n) = \left\{ \begin{array}{ll} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{array} \right\} = \sum_{d|n} \mu(d) \log \frac{n}{d}.$$

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Majorant: apply W-trick to $\Lambda_{\chi,R}^2$, appropriately normalized

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Open Problem (bounded gaps)

Prove there exist infinitely many 3-APs of primes with bounded common difference.

Maynard/Tao: $\exists$ infinitely many intervals of length k with $\gg \log k$ primes.

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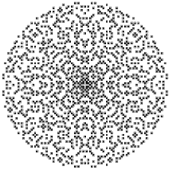
The Gaussian primes contain arbitrary constellations.



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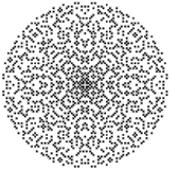
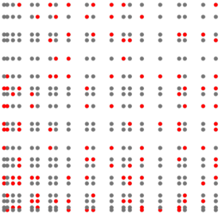


- The situation for dense subsets of $P \times P$ is quite different. See Tao–Ziegler & Cook–Magyar–Titichetrakun (also Fox–Z.)



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This gives another route for proving the relative Szemerédi theorem.

References

Conlon, Fox, Zhao
A relative Szemerédi theorem, 20pp

Zhao
An arithmetic transference proof of a relative Szemerédi thm, 6pp

Conlon, Fox, Zhao
The Green-Tao theorem: an exposition, 26pp