Equiangular Lines and Eigenvalue Multiplicities

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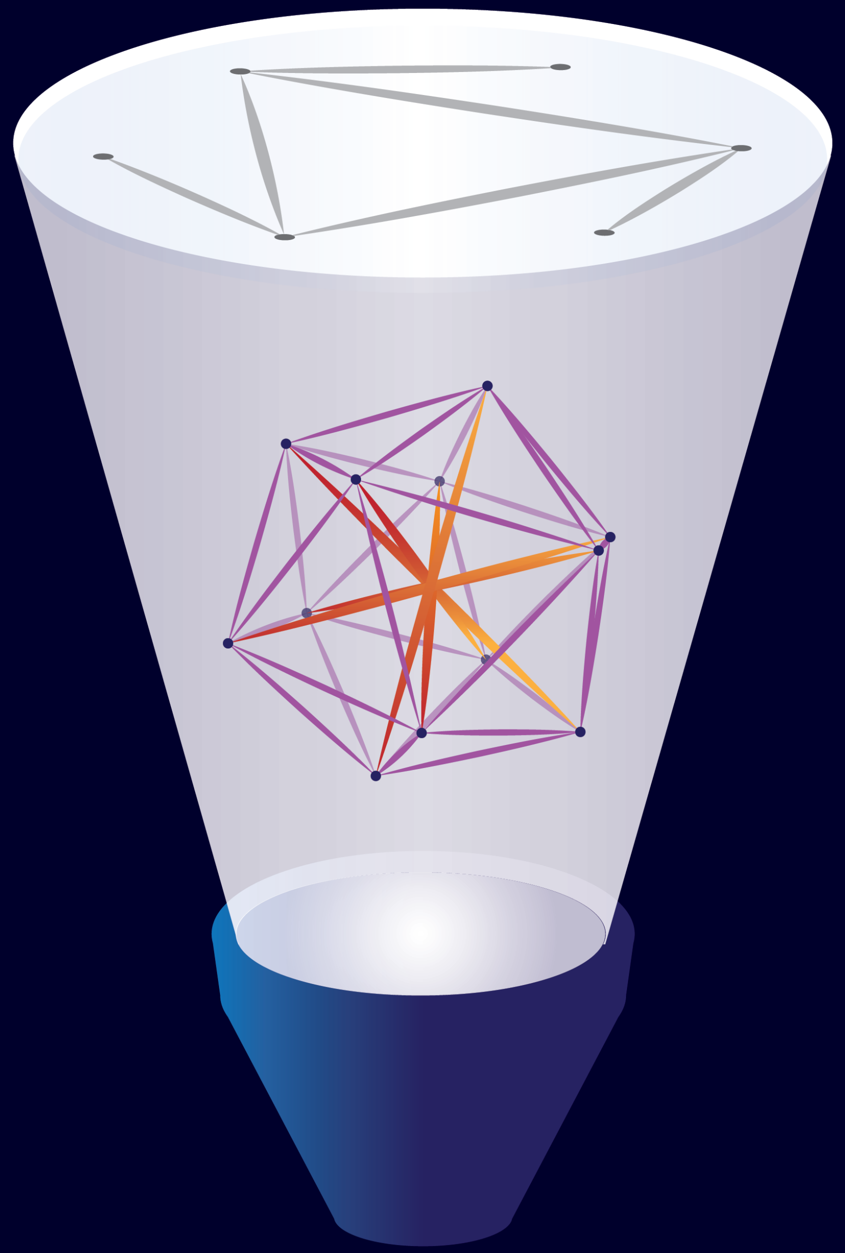
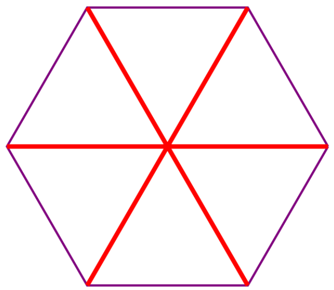


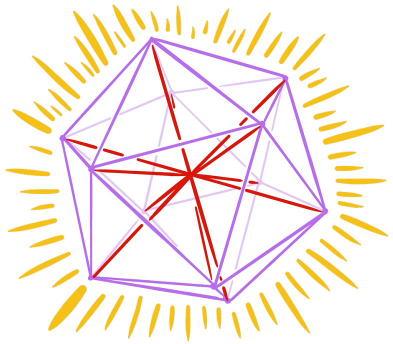
Image credit: Zilin Jiang

Equiangular Lines

$N(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise equal angles



$N(2) = 3$



$N(3) = 6$

Exact answer known for finitely many d

General bounds:

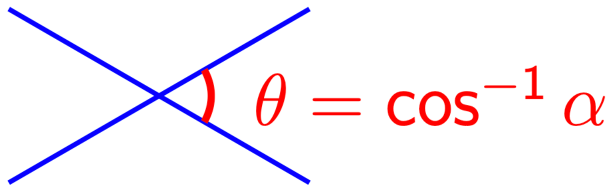
[de Caen '00] $$cd^2 \leq N(d) \leq \binom{d+1}{2}$$ [Gerzon '73]

In lower bound constructions, pairwise angles $\to 90°$ as $d \to \infty$

Equiangular Lines With a Fixed Angle

$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1}\alpha$

(focus: $\alpha > 0$ fixed, $d \to \infty$)



- $N_\alpha(d)$ grows linearly in *d*
  - in contrast to $N(d) = \Theta(d^2)$

- Problem: determine

$$\lim_{d\to\infty} \frac{N_\alpha(d)}{d}$$

Equiangular Lines With a Fixed Angle: History

[Lemmens, Seidel '73] $N_{1/3}(d) = 2(d-1) \quad \forall d \geq 15$

[Neumaier '89] $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for sufficiently large d

"the next interesting case will require substantially stronger techniques"

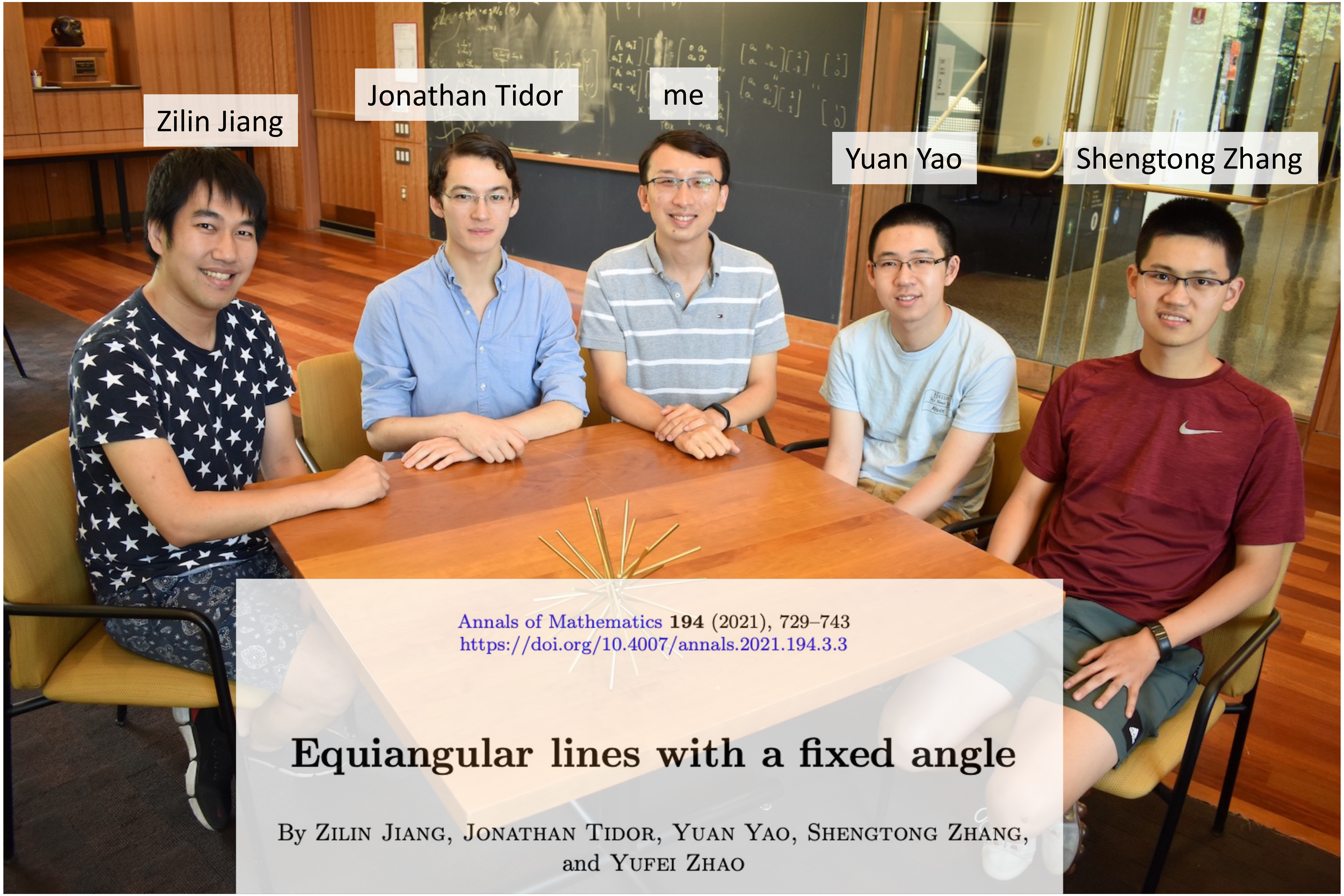
🕒🕔🕖🕘

[Bukh '16] $N_\alpha(d) \leq C_\alpha d$

[Balla, Dräxler, Keevash, Sudakov '18] $\limsup_{d\to\infty} N_\alpha(d)/d$ maximized at $\alpha = \frac{1}{3}$

[Jiang, Polyanskii '20] Determined $\lim_{d\to\infty} N_\alpha(d)/d \ \forall \alpha > 0.196$

[Jiang, Tidor, Yao, Zhang, Z. '21] Solved! Determined $\lim_{d\to\infty} N_\alpha(d)/d$ for all fixed α



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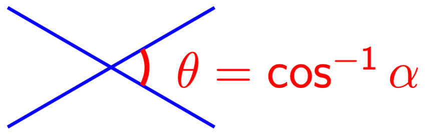
Equiangular lines with a fixed angle

By Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang, and Yufei Zhao

The Answer [Jiang, Tidor, Yao, Zhang, Z. '21]



$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1}\alpha$

For every integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

Other angles: $\forall$ fixed $\alpha \in (0,1)$, setting $\lambda = (1-\alpha)/(2\alpha)$ and

spectral radius order $k = k(\lambda)$

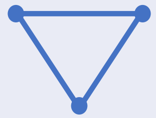
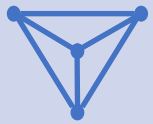
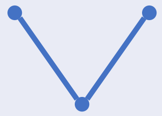
= min # vertices in a graph with top eigval exactly λ (adj. matrix)

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d)$ as $d \to \infty$

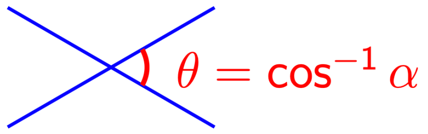
Spectral radius order

spectral radius order $k = k(\lambda)$
= min # vertices in a graph with top eigval exactly λ (adj. matrix)

Examples

α	λ	k	G
1/3	1	2	
1/5	2	3	
1/7	3	4	
$\frac{1}{1+2\sqrt{2}}$	$\sqrt{2}$	3	

The Answer [Jiang, Tidor, Yao, Zhang, Z. '21]



$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1} \alpha$

For every integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

Proof needs $d \geq 2^{2^{k^{1+o(1)}}}$
Conjecture: $\forall d \geq k^C$

Other angles: $\forall$ fixed $\alpha \in (0,1)$, setting $\lambda = (1-\alpha)/(2\alpha)$ and

spectral radius order $k = k(\lambda)$

= min # vertices in a graph with top eigval exactly λ (adj. matrix)

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d)$ as $d \to \infty$

- $o(d)$ is $O_\alpha(d / \log\log d)$
- Jiang–Polyanskii '20 conj: $O_\alpha(1)$
- No! Carl Schildkraut ('23+) constructed $\Omega_\alpha(\log\log d)$

Second Eigenvalue Multiplicity

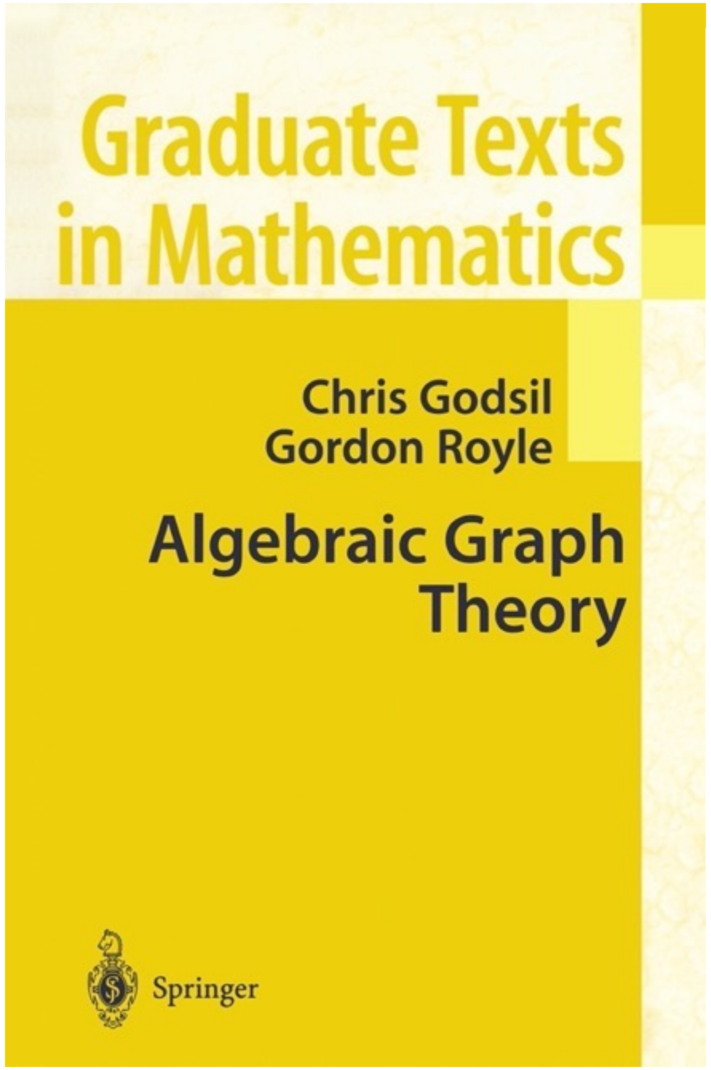
A connected bounded degree graph has sublinear second eigenvalue multiplicity
(always referring to the adjacency matrix)

Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21)

A connected n-vertex graph with maximum degree Δ has second largest eigenvalue with multiplicity

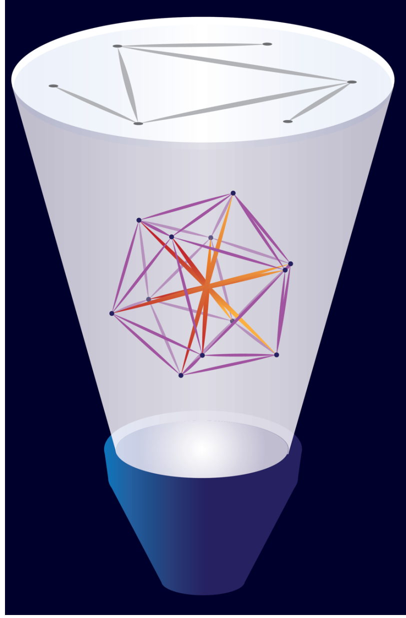
$$\leq C \log \Delta \frac{n}{\log \log n}$$

Connection to Spectral Graph Theory

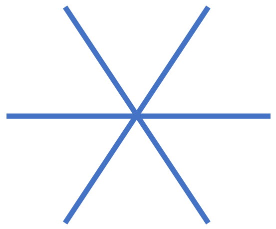


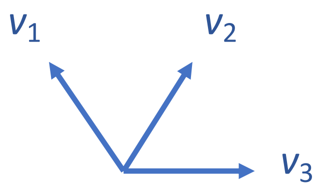
The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In $\mathbb{R}^d$, for example, a simplex contains at most $d + 1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of $\mathbb{R}^d$, and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in $\mathbb{R}^d$ such that the angle between any two distinct lines is the same. We call this a set of *equiangular lines*. In this chapter we show how the problem of determining the maximum number of equiangular lines in $\mathbb{R}^d$ can be expressed in graph-theoretic terms.

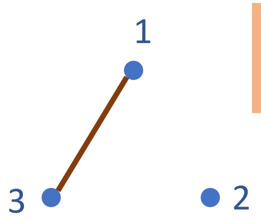
Connection to Spectral Graph Theory



Equiangular lines in $\mathbb{R}^d$ ⟶ unit vectors in $\mathbb{R}^d$ ⟶ graph G







Edge = obtuse
Non-edge = acute

Given a list of vectors $v_1, \dots, v_n \in \mathbb{R}^d$, Gram matrix is PSD and rank ≤ d:

$$\text{Gram matrix} = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix} = (1-\alpha)I - 2\alpha A_G + \alpha J$$

J = all 1s matrix

Equivalent problem: given α, d, find graph G with max # vertices N s.t.

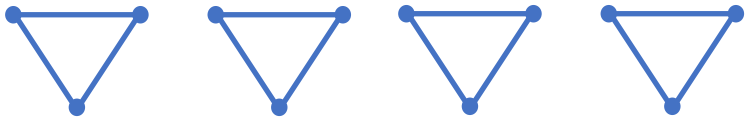
$(1-\alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank ≤ d.

Connection to Spectral Graph Theory

Problem: Given α, d, find graph G with max # vertices N s.t.
$\mathrm{Gram} = (1-\alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank ≤ d.

Example: recall $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ for all large d.

To verify $N_{1/5}(9) \geq 12$, check

G = 

$$(1-\alpha)I - 2\alpha A_G + \alpha J = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
-\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\hline
\alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & -\alpha & -\alpha & 1 & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\hline
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha & \alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & -\alpha & 1 & \alpha & \alpha & \alpha \\
\hline
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & 1 & -\alpha & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & 1 & -\alpha \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & -\alpha & -\alpha & 1
\end{array}\right)$$

is PSD and rank 9
$(\alpha = 1/5)$

Upper Bound on *N*

Problem: Given α, d, find graph G with max # vertices N s.t.

$\text{Gram} = (1-\alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank ≤ d.

By rank–nullity

$$\begin{aligned} N &= \operatorname{rank}(\text{Gram}) + \operatorname{nullity}(\text{Gram}) \\ &\le d + \operatorname{null}((1-\alpha)I - 2\alpha A_G + \alpha J) \\ &\le d + \operatorname{null}\big((1-\alpha)I - 2\alpha A_G\big) + 1 \end{aligned}$$

Multiplicity of $\lambda = \frac{1-\alpha}{2\alpha}$ as an eigval of A_G

Since Gram is PSD, if $\lambda = \frac{1-\alpha}{2\alpha}$ is an eigval of A_G, it must be either

the largest eigval (equality case) or 2nd largest (need to rule out)

easy *hard*

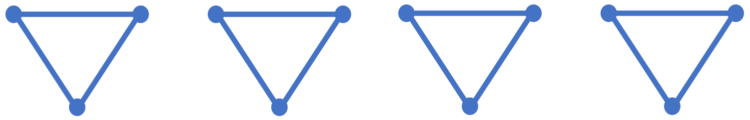
Recap

- Equiangular lines in $\mathbb{R}^d$ → unit vectors in $\mathbb{R}^d$ → graph G

  N = # lines # vertices

- $N \le d + \mathrm{mult}(\lambda, A_G) + 1$
- Optimal configuration (for large d) turns out to be

  G = disjoint copies of a fixed graph with top eigval exactly $\lambda = \frac{1-\alpha}{2\alpha}$



- What happens if λ is the 2nd eigval of G ?
  - Can assume G is connected from now on
- Want to show that $\mathrm{mult}(\lambda_2, A_G)$ is small

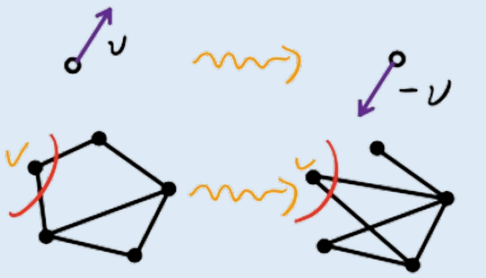
Second Eigenvalue Multiplicity

Q: must all connected graphs have small 2nd eigval multiplicity?

No. k-clique has eigvals $k-1$ (once) and -1 ($k-1$ times)

Not all graphs can arise from equiangular lines

Switching operation:



Theorem (Balla, Dräxler, Keevash, Sudakov '18)

$\forall \alpha\ \exists \Delta = \Delta(\alpha)$: can switch so that max degree ≤ Δ

[Balla '21+] $\Delta = O(\alpha^{-4})$ & tight

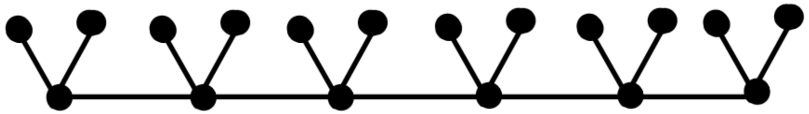
Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21) A connected n-vertex graph with max deg ≤ Δ has 2nd eigval multiplicity $O_\Delta\left(\frac{n}{\log \log n}\right)$

Sublinear Second Eigenvalue Multiplicity

Theorem. (Jiang, Tidor, Yao, Zhang, Z. ’21) A connected *n*-vertex graph with max deg ≤ Δ has 2nd eigval multiplicity $O_\Delta\left(\frac{n}{\log\log n}\right)$

More generally, jth eigval multiplicity $O_{\Delta,j}\left(\frac{n}{\log\log n}\right)$ for fixed j

Near miss examples

- Strongly regular graphs (e.g., complete graph, Paley graph)
  - Not bounded degree
-  has eigval 0 with linear multplicity
  - 0 is not the 2nd largest eigval
-  has top eigval with linear multiplicity
  - not connected

Second Eigenvalue Multiplicity

Open: Maximum 2nd eigval mult of conn. bounded degree graph on *n* vertices?

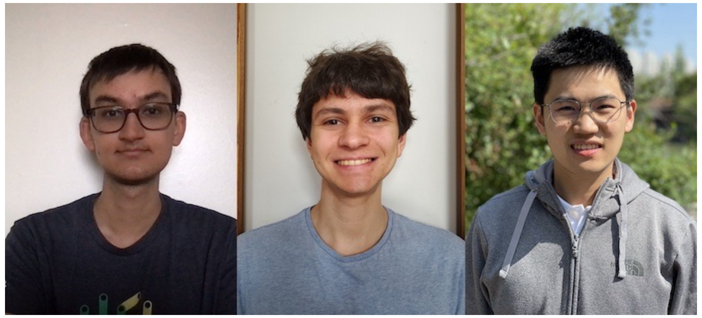
- [Jiang, Tidor, Yao, Zhang, Z. '21]

$$\mathrm{mult}(\lambda_2, G) \le \frac{C_\Delta n}{\log\log n}$$

- [Haiman, Schildkraut, Zhang, Z. '22]
  ∃ infinite family of bounded degree graphs with

$$\mathrm{mult}(\lambda_2, G) \ge \sqrt{\frac{n}{\log_2 n}}$$

Construction.
Step 1. Cayley graph on $\mathrm{Aff}(\mathbb{F}_q)$ generated by:
(I) a multiplicative generator, and
(II) an additive shift.
Step 2. Each (II) edge ⇝ path of length log *q*



Milan Haiman | Carl Schildkraut | Shengtong Zhang

  - Eigenvalues tend not to collide "by accident"
  - Relies on group representations to get multiple eigval. Barrier at $\sqrt{n}$

- **Open:** $\mathrm{mult}(\lambda_2, G) < n^{1-c}$? (⇒ equiangular lines theorem for dimension $d > k^C$)

Second Eigenvalue Multiplicity

Q: Maximum 2nd eigval multiplicity of connected bounded degree *n*-vertex graph?

[Jiang, Tidor, Yao, Zhang, Z. '21] $\mathrm{mult}(\lambda_2, G) = O(n/\log\log n)$

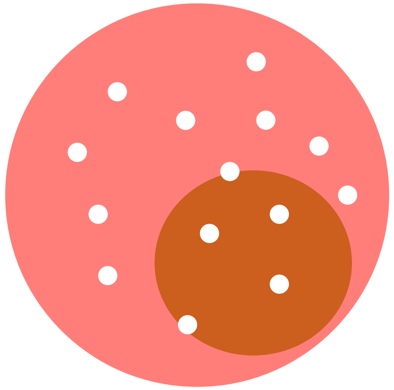
- For expander graphs ($N(A) \geq (1+\delta)|A| \ \ \forall |A| \leq n/2$),
  $$\mathrm{mult}(\lambda_2, G) = O(n/(\log n)^c)$$
- [Lee–Makarychev '08, building on Gromov, Colding–Minicozzi, Kleiner]
  For non-expanding Cayley graphs, $\mathrm{mult}(\lambda_2, G) = O(1)$
  - Examples of non-expanding: Abelian groups, nilpotent groups of bounded step
- [McKenzie, Rasmussen, Srivastava '21] For regular graphs
  $$\mathrm{mult}(\lambda_2, G) = O(n/(\log n)^c)$$
  - A typical length 2*k* closed walk covers $\geq k^c$ vertices
- [Haiman, Schildkraut, Zhang, Z. '22] Lower bound constructions
  Irregular: $\gtrsim \sqrt{n/\log n}$ Cayley: $\gtrsim n^{2/5}$

Proof Sketch

Theorem. $\mathrm{mult}(\lambda_2, G) \le C_\Delta n / \log\log n$ for connected G

$r = c \log\log n$ $s = c \log n$ $\lambda = \lambda_2(G)$

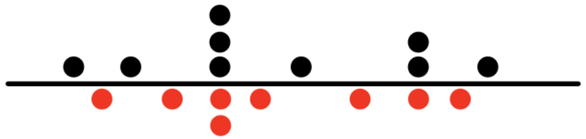
vertex removal lowers spectral radius



- H = G with a small r-net removed
- radius s balls in H typ. have spectral radius $< \lambda - \varepsilon$
  - By counting length $2s$ closed walks
- Bound 2nd eigval multiplicity in H via moments:

$$\mathrm{mult}(\lambda, H)\lambda^{2s} \le \sum_i \lambda_i(H)^{2s} = \mathrm{tr} A_H^{2s} = \#\text{closed length } 2s \text{ walks in } H$$

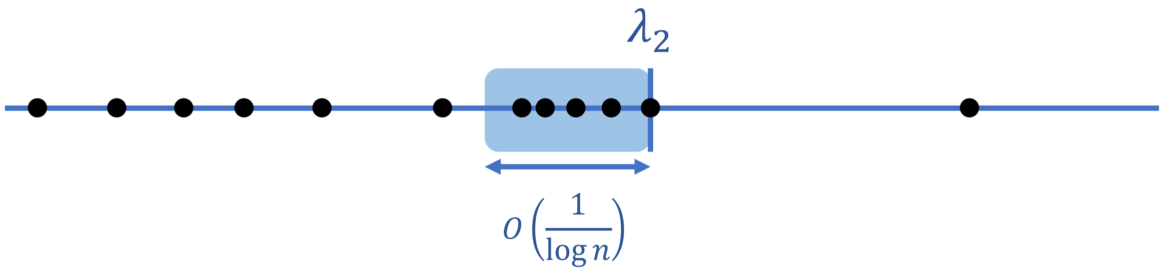
$$\le \sum_{v \in V(H)} \lambda_1(\text{radius } s \text{ ball around } v \text{ in } H)^{2s} \le n(\lambda - \varepsilon)^{2s}$$



- $\Rightarrow \mathrm{mult}(\lambda, H) = o(n)$
- $\Rightarrow \mathrm{mult}(\lambda, G) \le \mathrm{mult}(\lambda, H) + |\mathrm{net}| = o(n)$ by Cauchy eigval interlacing

Approximate 2nd Eigenvalue Multiplicity

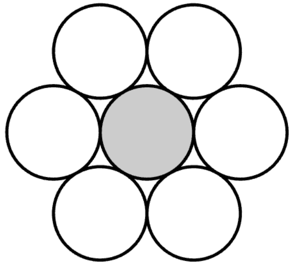
- Proof also bounds the "approximate 2nd eigval multiplicity", showing at most $O\left(\frac{n}{\log\log n}\right)$ eigenvalues (incl. mult.) within $O\left(\frac{1}{\log n}\right)$ of λ_2



- [Haiman, Schildkraut, Zhang, Z. '22]
  A construction showing the above bounds are tight
  - Demonstrates a limitation of the trace method

Spherical Codes

- $L \subseteq [-1,1)$. An *L*-code in $\mathbb{R}^d$ is a set of unit vectors whose pairwise inner products lie in *L*
- $N_L(d)$ = size of largest *L*-code in $\mathbb{R}^d$
- Points on a sphere with pairwise angle $\geq \theta$: $L = [-1, \cos\theta]$
  - Kissing number $\mathbb{R}^d$: $L = \left[-1, \frac{1}{2}\right]$
  - Sphere packing upper bounds in high dimensions
  - Linear programming bound (Delsarte ’73)



- **Equiangular lines:** $L = \{-\alpha, \alpha\}$

Beyond linear programming bounds …

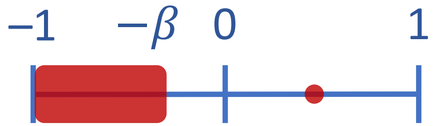
Spherical Codes

$N_L(d)$ = size of largest L-code in $\mathbb{R}^d$, i.e., unit vec, pairwise inner products lie in L

- [Bukh '05]

  For fixed $L = [-1, -\beta] \cup \{\alpha\}$ with $\beta > 0$

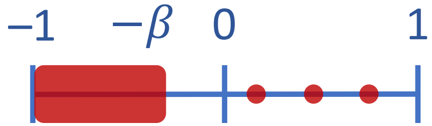
  $$N_L(d) = O_L(d)$$



- [Balla, Dräxler, Keevash, Sudakov '18]

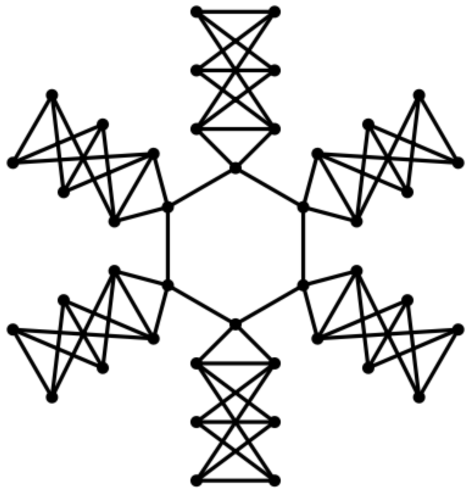
  For fixed $L = [-1, -\beta] \cup \{\alpha_1, \dots, \alpha_k\}$ with $\beta > 0$

  $$N_L(d) = O_L(d^k)$$



Further research: determine $N_L(d)$ more precisely

Spherical Two-Distance Sets



[Jiang, Tidor, Yao, Zhang, Z. ’20+] [Jiang, Polyanskii ’21+]

- **Problem.** For fixed $\alpha, \beta > 0$, determine

$$\lim_{d\to\infty} \frac{N_{\{-\beta,\alpha\}}(d)}{d}$$

- A conjectural limit in terms of eigenvalue of signed graphs
- Solved in special cases: $\alpha < 2\beta$ or $(1-\alpha)/(\alpha+\beta) < 2.019\cdots$
- Open in general
- **Obstacle:** Sublinear eigenvalue multiplicity FALSE for signed graphs
  - E.g., ∃ bounded degree graph whose most negative eigval multiplicity is linear

Solution Framework

I. Forbidden local configurations / subgraphs
   - Using Gram matrix is PSD

II. Global graph structure
   - Graph theory, Ramsey theory
   - Equiangular lines: bounded degree graph
   - Spherical two-distance sets: complete multipartite XOR bounded degree

III. New insights in spectral graph theory ??
   - [JTYZZ] Sublinear eigenvalue multiplicity in connected bounded degree graphs
   - [Jiang Polyanski ’21+] {signed graphs with largest eigenvalue $\leq \lambda$} is characterized by forbidding a finite set of induced subgraphs iff $\lambda < 2.019\cdots$

Complex Equiangular Lines

Unrestricted angles

- **Zauner's conjecture**: $N^{\mathbb{C}}(d) = d^2$ for all *d* (known: $N^{\mathbb{C}}(d) \leq d^2$)
  i.e., $\exists\ d^2$ unit vec. in $\mathbb{C}^d$ with equal abs. of pairwise inner product
  - "SIC-POVM" from quantum mechanics
  - Verified in small dim. (exactly for $d \leq 53$, numerically for $d \leq 193$)

Restricted angles

- Determine $\lim_{d\to\infty} N_{\alpha}^{\mathbb{C}}(d)/d$

Equiangular subspaces in $\mathbb{R}^d$

- Configs of *k*-dim. subspaces in $\mathbb{R}^d$ with given pairwise angles

Equiangular lines and eigenvalue multiplicities

Equiangular lines with a fixed angle.

$N_\alpha(d)$ = max # of lines in $\mathbb{R}^d$ with pairwise angle $\cos^{-1}\alpha$

$\forall$ integer $k \geq 2$, $N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor$ $\forall d \geq d_0(k)$

Other angles: $\forall$ fixed $\alpha \in (0,1)$, setting $\lambda = (1-\alpha)/(2\alpha)$

spectral radius order $k = k(\lambda)$

= min # vertex in a graph with top eigval exactly λ

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor$ $\forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d)$ as $d \to \infty$

Sublinear eigenvalue multiplicity of bounded degree graphs.

A connected n-vertex graph with maximum degree Δ has second largest eigenvalue with multiplicity $\leq C \log \Delta \dfrac{n}{\log \log n}$