THE COEFFICIENTS OF A TRUNCATED FIBONACCI POWER SERIES

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Abstract

In this note, we give a short proof of the fact that the coefficients of the polynomial

$$A_n(x) = (1-x)(1-x^2)(1-x^3)\cdots(1-x^{F_n})(1-x^{F_{n+1}})$$

are all equal to $-1, 0$ or 1, where F_n is the n-th Fibonacci number. This improves the previous result that the coefficients of $\prod_{n\geq 2}\left(1-x^{F_n}\right)$ are all equal to $-1, 0$ or 1.

Consider the infinite product

$$\begin{aligned} A(x) &= \prod_{n\geq 2}\left(1-x^{F_n}\right) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots \\ &= 1 - x - x^2 + x^4 + x^7 - x^8 + x^{11} - x^{12} - x^{13} + x^{14} + x^{18} + \cdots . \end{aligned}$$

regarded as a formal power series, where F_n is the n-th Fibonacci number. There is a very simple combinatorial interpretation of the coefficients of $A(x)$, namely, the coefficient of x^m is $r_E(m) - r_O(m)$, where $r_E(m)$ (resp. $r_O(m)$) is the number of ways to write m as a sum of an even (resp. odd) number of distinct positive Fibonacci numbers. Robbins [2] showed that the coefficients of $A(x)$ are all equal to $-1, 0$ or 1, and Ardila [1] gave a simple recursive description of the coefficients of $A(x)$.

In this note, we give a short proof of a somewhat stronger result. Namely, we show that any partial product of $A(x)$, considered as a polynomial, also has coefficients $-1, 0, 1$.

Proposition 1. Let n be a positive integer. The coefficients of the polynomial

$$A_n(x) = (1-x)(1-x^2)(1-x^3)\cdots(1-x^{F_n})(1-x^{F_{n+1}})$$

are all equal to $-1, 0$ or 1.

For instance, the first few partial products are

$$\begin{aligned} A_1(x) &= 1 - x \\ A_2(x) &= 1 - x - x^2 + x^3 \\ A_3(x) &= 1 - x - x^2 + x^4 + x^5 - x^6 \\ A_4(x) &= 1 - x - x^2 + x^4 + x^7 - x^9 - x^{10} + x^{11} \\ A_5(x) &= 1 - x - x^2 + x^4 + x^7 - x^8 + x^{11} - x^{12} - x^{15} + x^{17} + x^{18} - x^{19} \\ A_6(x) &= 1 - x - x^2 + x^4 + x^7 - x^8 + x^{11} - x^{12} - x^{13} + x^{14} + x^{18} - x^{19} \\ &\quad - x^{20} + x^{21} - x^{24} + x^{25} + x^{28} - x^{30} - x^{31} + x^{32} \end{aligned}$$

Combinatorially, this is equivalent to saying that if we are only allowed to use distinct parts taken from the set $\{F_2, F_3, \ldots, F_n\}$, then the number of partitions of m into an odd number of parts differs by at most one from the number of partitions of m into an even number of parts.

Note that Proposition 1 implies the result that the coefficients of $A(x)$ are $-1, 0$ and 1, since the terms of $A_n(x)$ agree with $A(x)$ until at least up to the term $x^{F_{n+1}-1}$. Thus, by choosing n arbitrarily large, our result implies the result about $A(x)$.

Proof of Proposition 1. We say that a polynomial is *timid* if each of its coefficients is $-1, 0$ or 1. Let us construct the auxiliary polynomials

$$\begin{aligned} B_n(x) &= (1-x)(1-x^2)(1-x^3)\cdots(1-x^{F_n})(1-x^{F_{n+1}}-x^{F_{n+2}}), \\ \text{and}\quad C_n(x) &= (1-x)(1-x^2)(1-x^3)\cdots(1-x^{F_n})(1+x^{F_n}-x^{F_{n+2}}). \end{aligned}$$

In the $n=1$ case, we define $B_1(x) = 1 - x^{F_2} + x^{F_3}$ and $C_1 = 1 + x^{F_1} - x^{F_3}$. We will show by induction that the polynomials A_n, B_n, C_n are all timid for all positive integer n.

We can check the base cases ($n = 1, 2$) manually. Now suppose that we know that A_k, B_k, C_k are all timid for all $k < n$. We want to prove that A_n, B_n, C_n are all timid as well.

First, we show that A_n is timid. We have

$$\begin{aligned} A_n(x) &= A_{n-3}(x)(1-x^{F_{n-1}})(1-x^{F_n})(1-x^{F_{n+1}}) \\ &= A_{n-3}(x)(1-x^{F_{n-1}}-x^{F_n}-x^{F_{n+1}}+x^{F_{n-1}+F_n}+x^{F_{n-1}+F_{n+1}}+x^{F_n+F_{n+1}}-x^{F_{n-1}+F_n+F_{n+1}}) \\ &= A_{n-3}(x)(1-x^{F_{n-1}}-x^{F_n}+x^{F_{n-1}+F_{n+1}}+x^{F_{n-2}+F_{n-1}+F_{n+1}}-x^{F_{n-1}+F_n+F_{n+1}}) \\ &= A_{n-3}(x)(1-x^{F_{n-1}}-x^{F_n})+x^{F_{n-1}+F_{n+1}}A_{n-3}(x)(1+x^{F_{n-2}}-x^{F_n}) \\ &= B_{n-2}(x)+x^{F_{n-1}+F_{n+1}}C_{n-2}(x). \end{aligned}$$

Now, notice that the degree of $B_{n-2}(x)$ is $F_2 + F_3 + \cdots + F_{n-2} + F_n = 2F_n - F_3$, and we have $F_{n-1}+F_{n+1} > 2F_n - F_3$ since $(F_{n-1}+F_{n+1}) - (2F_n - F_3) = F_{n-3} + F_3 > 0$. Informally, this means that when we add the two polynomials $B_{n-2}(x)$ and $x^{F_{n-1}+F_{n+1}}C_{n-2}(x)$, the terms "don't mix." Then, the fact that B_{n-2} and C_{n-2} are both timid implies that $A_n(x) = B_{n-2}(x) + x^{F_{n-1}+F_{n+1}}C_{n-2}(x)$ is timid as well.

Next, we show that B_n is timid. We have

$$\begin{aligned} B_n(x) &= A_{n-2}(x)(1-x^{F_n})(1-x^{F_{n+1}}-x^{F_{n+2}}) \\ &= A_{n-2}(x)(1-x^{F_{n+1}}-x^{F_{n+2}}-x^{F_n}+x^{F_n+F_{n+1}}+x^{F_n+F_{n+2}}) \\ &= A_{n-2}(x)(1-x^{F_n}-x^{F_{n+1}})+A_{n-2}(x)x^{F_n+F_{n+2}} \\ &= B_{n-1}(x)+A_{n-2}(x)x^{F_n+F_{n+2}}. \end{aligned}$$

Now we argue as before. Since the degree of B_{n-1} is $2F_{n+1} - F_3$, which is less than $F_n + F_{n+2}$, the fact that B_{n-1} and A_{n-2} are both timid implies that B_n is timid as well.

Finally, we show that C_n is timid. We have

$$\begin{aligned} C_n(x) &= A_{n-2}(x)(1-x^{F_n})(1+x^{F_n}-x^{F_{n+2}}) \\ &= A_{n-2}(x)(1+x^{F_n}-x^{F_{n+2}}-x^{F_n}-x^{2F_n}+x^{F_n+F_{n+2}}) \\ &= A_{n-2}(x)(1-x^{2F_n+F_{n-1}}-x^{2F_n}+x^{2F_n+F_{n+1}}) \\ &= A_{n-2}(x)-x^{2F_n}A_{n-2}(x)(1+x^{F_{n-1}}-x^{F_{n+1}}) \\ &= A_{n-2}(x)-x^{2F_n}C_{n-1}(x). \end{aligned}$$

The degree of A_{n-2} is $F_2 + F_3 + \cdots + F_{n-1} = F_{n+1} - F_3$, and $F_{n+1} - F_3$ is less than $2F_n$ since $2F_n - (F_{n+1} - F_3) = F_{n-2} + F_3 > 0$. Therefore, since A_{n-2} and C_{n-1} are both timid, C_n is timid as well.

We have completed our proof that A_n, B_n, C_n are timid for all n. The proposition follows. □

This proof also allows us to make a slightly more general conclusion.

Proposition 2. Let $t_1, t_2, \ldots$ be a sequence of positive integers satisfying $t_1 < t_2$ and $t_{n+2} = t_{n+1} + t_n$ for all positive integers n. Then the coefficients of the polynomial $(1 - x^{t_1})(1 - x^{t_2}) \cdots (1 - x^{t_n})$ are all equal to $-1, 0$ or 1 for all positive integers n.

To prove Proposition 2, we simply have to replace every occurrence of F_n with t_{n-1} in the proof of Proposition 1.

For instance, for any positive integers $m < n$, the coefficients of polynomials $\prod_{k=m+1}^{n} \left(1 - x^{F_k}\right)$ and $\prod_{k=m}^{n} \left(1 - x^{L_k}\right)$ are all equal to $-1, 0$ or 1. Here L_k is the k-th Lucas number.

Note that while $\prod_{n \geq 1}(1 - x^n)$ has coefficients $-1, 0$ and 1 due to Euler's pentagonal number theorem, we cannot say the same thing about its partial products, as $\prod_{n=1}^{4}(1 - x^n) = 1 - x - x^2 + 2x^5 - x^8 - x^9 + x^{10}$. Also, if a sequence of positive integers (t_n) satisfies $t_{n+1} > t_n + t_{n-1} + \cdots + t_1$ for all n, then the polynomial $\prod_{k=1}^{n}(1 - x^{t_k})$ clearly always has coefficients $-1, 0$ or 1. It would be interesting to characterize all sequences that have similar properties.

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References

[1] F. Ardila. "The Coefficients of a Fibonacci Power Series." *The Fibonacci Quarterly* **42.3** (2004): 202-204.

[2] N. Robbins. "Fibonacci Partitions." *The Fibonacci Quarterly* **34.4** (1996): 306-313.

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