Recap

Key incidence estimates (Guth-Katz)

$\mathcal{L}$: L lines in $\mathbb{R}^3$

(a) If $\leq \sqrt{L}$ lines in any plane or deg 2 surface, then $|P_2(\mathcal{L})| \lesssim L^{3/2}$.

(b) If $\leq \sqrt{L}$ lines in any plane, then $|P_r(\mathcal{L})| \lesssim \frac{L^{3/2}}{r^2}$ $\forall 3 \leq r \leq 2\sqrt{L}$

Last time we proved (b) for $r=3$

Szemerédi-Trotter in $\mathbb{R}^2$

$\mathcal{L}$: L lines in $\mathbb{R}^2$

$$|P_r(\mathcal{L})| \lesssim \frac{L^2}{r^3} + \frac{L}{r}$$

local to global

Plane detection lemma

$\forall$ polynomials P in $\mathbb{R}^3$, there is a list of 9 polynomials SP s.t.

(1) If $x \in Z(P)$, then $SP(x) = 0 \iff x$ is critical or flat

(2) If x is contained in 3 lines in $Z(P)$ then $SP(x) = 0$

(3) $\deg SP \leq 3 \deg P$

(4) If P is irreducible and SP vanishes on $Z(P)$, and $Z(P)$ contains a regular pt, then $Z(P)$ is a plane.

2-rich points

<u>Thm</u> (GK) $\mathcal{L}$: L lines in $\mathbb{R}^3$ or $\mathbb{C}^3$, $\le B$ lines in any plane or deg 2 surface then

$$|P_2(\mathcal{L})| \lesssim LB + L^{3/2}$$

<u>Recall</u> Regulus (eg. $z = xy$)

L lines can make $\frac{1}{4}L^2$ intersections

$\frac{L}{B}$ planes/reguli, B lines on each forming $\frac{1}{4}B^2$ 2rich pts

<u>Rmk</u> When $B = L$, no good example.

<u>Classification</u> of doubly ruled surfaces (19th century).

<u>Thm</u>. $P \in \mathrm{Poly}(\mathbb{C}^3)$ irred.

If $Z(P)$ doubly-ruled (every pt on $Z(P)$ is contained in 2 lines in $Z(P)$) then $\deg P = 1$ or 2

$Z(P)$ plane or regulus.

<u>Rmk</u> GK thm is a strong quantitative strengthening of this classification

Flecnode

$z \in Z(P)$ is flecnodal if P vanishes at z to third order in some direction.

If $z \in Z(P)$ lies on a line in $Z(P)$, then it's flecnodal.

Salmon's Flecnodal polynomial

$\text{Flec}\,P \in \text{Poly}(\mathbb{C}^3)$

- $\deg \text{Flec}\,P \leq 11 \deg P$
- $z \in Z(P)$ is flecnodal $\Leftrightarrow \text{Flec}\,P(z) = 0$.

Local-to-global principle

Thm (Monge-Cayley-Salmon) If $P \in \text{Poly}(\mathbb{C}^3)$ irred every pt on $Z(P)$ is flecnodal, then $Z(P)$ is ruled.

Thm (GK) If $P \in \text{Poly}(\mathbb{C}^3)$ irred, every pt of $Z(P)$ is doubly flecnodal, then $\deg P = 1$ or 2

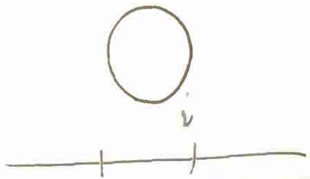
$Z(P)$ is plane or regulus.

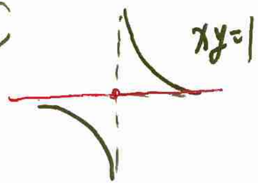
Furthermore, there is a list of $O(1)$ polynomials of deg $O(\deg P)$ that detects whether $z \in Z(P)$ is doubly flecnodal.

Elimination theory

Projection theory

$\mathbb{C}$





Thm (Guth-Katz).

$\mathcal{S}$: S pts, $\mathcal{L}$: L lines in $\mathbb{R}^3$

$\le B$ lines of $\mathcal{L}$ in any plane.

$B \geq \sqrt{L}$

Then

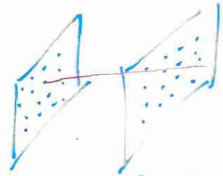
$$I(\mathcal{S},\mathcal{L}) \lesssim S^{1/2} L^{3/4} + B^{1/3} L^{1/3} S^{2/3} + L + S.$$

Thm Szemerédi-Trotter. in $\mathbb{R}^2$

$$I(\mathcal{S},\mathcal{L}) \lesssim S^{2/3} L^{2/3} + S + L.$$

Examples:

1st term



all lines joining one pt from each

$(a,b,0)$ $(a,b,1)$

$1 \le a,b \le L^{1/4}$

2nd term.

$\frac{L}{B}$ planes and B lines on each plane, and use grid example in ST.

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Polynomial partitioning.

P $\deg \le D$

$\mathbb{R}^2 \setminus Z(P)$ into cells

of $\lesssim S/D^2$ pts

$$I(\mathcal{S},\mathcal{L}) = I(\mathcal{S}_{cell}, \mathcal{L}) \quad \text{simple bound aggregate over cells}$$

$$+\, I(\mathcal{S}_{alg}, \mathcal{L}_{cell}) \quad \text{each } \ell \in \mathcal{L}_{cell} \text{ meets } Z(P) \text{ at} \le D \text{ pts}$$

$$+\, I(\mathcal{S}_{alg}, \mathcal{L}_{alg}) \quad |\mathcal{L}_{alg}| \le D$$

For $\mathbb{R}^3$, apply similar strategy

Control $I(\mathcal{S}_{alg}, \mathcal{L}_{alg})$

Divide $Z(P)$ into planar & non-planar parts

lines of $\mathcal{L}$ in same plane $\subset Z(P)$

$$I(\mathcal{S}_{alg},\mathcal{L}_{alg}) = \underbrace{I(\mathcal{S}_{alg}, \mathcal{L}_{pl})}_{\substack{\text{Apply S-T.} \\ \text{Use} \le B \text{ lines in} \\ \text{each plane}}} + I(\mathcal{S}_{alg}, \mathcal{L}_{alg} \setminus \mathcal{L}_{pl})$$

Special = flat curvature

$$I(\mathcal{S}_{alg}, \mathcal{L}_{alg} \setminus \mathcal{L}_{pl})$$

$$\le I(\mathcal{S}_{nonsp}, \mathcal{L}_{alg}) \le 2 S_{nonsp} \quad \text{by (2) of plane detecting lem}$$

$$+\, I(\mathcal{S}_{sp}, \mathcal{L}_{nonsp}) \le 3DL \quad \text{by (3)}$$

$$+\, I(\mathcal{S}_{sp}, \underbrace{\mathcal{L}_{sp} \setminus \mathcal{L}_{pl}}_{[illegible]})$$