Last time

Key incidence estimates (Guth-Katz)

$\mathcal{L}$: L lines in $\mathbb{R}^3$

(a) If $\leq \sqrt{L}$ lines lie in any plane or deg 2 surface, then $|P_2(\mathcal{L})| \lesssim L^{3/2}$

(b) If $\leq \sqrt{L}$ lines lie in any plane, then $|P_r(\mathcal{L})| \lesssim \frac{L^{3/2}}{r^2}$ $\forall 3 \leq r \leq 2\sqrt{L}$

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degree reduction.

We proved:

Prop $\mathcal{L}$: L lines in $\mathbb{F}^3$. Each line of $\mathcal{L}$ contains $\geq A$ pts of $P_2(\mathcal{L})$. Then $\deg(\mathcal{L}) \lesssim \frac{L}{A}$

Combinatorial structure

$\Big\downarrow$ degree reduction: small deg poly vanishing on $\mathcal{L}$

Algebraic structure

$\Big\downarrow$ $Z(P)$ has many flat points

Geometric structure

Planar clustering

Thm $\exists$ const K.
$\mathcal{L}$: L lines in $\mathbb{R}^3$. If each line contains $\geq A = KL^{1/2}$ pts of $P_3(\mathcal{L})$, then $\mathcal{L}$ lies in $\leq KL/A$ planes.

Cor $\mathcal{L}$: L lines in $\mathbb{R}^3$, $\leq B$ lines of $\mathcal{L}$ in any plane. If $B \geq \sqrt{L}$, then

$$|P_3(\mathcal{L})| \lesssim BL.$$

(Induction on L to deduce Cor from Thm)

Degree reduction

- P min deg nonzero poly vanishing on $\mathcal{L}$
- $\deg P \lesssim L/A$
- Goal: show that P is a product of linear factors ($\Rightarrow Z(P)$ is a union of $\lesssim \frac{L}{A}$ planes)

- $x \in P_3(\mathcal{L})$
  - If 3 lines not coplanar (joint)
    x is a critical point of P
  - If 3 lines are coplanar
    $Z(P)$ is flat at x

Lem Every pt of $P_3(\mathcal{L})$ is either a critical point or a flat point of $Z(P)$.

Being critical / flat is contagious b/c it's a low-deg polynomial condition.

<u>Lem</u> For any $P \in Poly_D(\mathbb{R}^3)$, there is a list of 9 polynomials

$$SP_1, SP_2, \dots, SP_9 \in Poly_{3D}(\mathbb{R}^3)$$

s.t. $x \in Z(P)$ is critical or flat

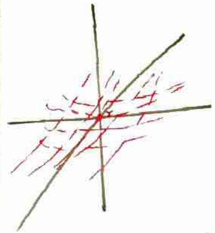
$$\iff SP_1(x) = SP_2(x) = \dots = SP_9(x) = 0$$

Contagious: if $\ell \in \mathcal{L}$ contains $\geq A > 3D$ of $P_3(\mathcal{L})$, then SP_j vanish on ℓ, so ℓ is critical/flat.

<u>Flat points</u> Two equivalent definitions

<u>Def 1.</u> Smooth manifold $\underset{\substack{\rotatebox{90}{$\in$} \\ x}}{M^2} \subset \mathbb{R}^3$

By chg of coord, assume $x = 0$, and tangent plane at x is $x_3 = 0$.

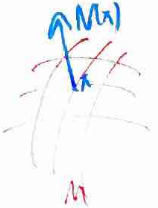


Locally near 0, M is described by $x_3 = h(x_1, x_2)$

$h(0) = 0$, $\nabla h(0) = 0$

x is <u>flat</u> iff the second derivatives of h vanish.

Def 2. $N : M \to S^2$ unit normal vector



x is **flat** if the derivatives of N vanish at x.

i.e. $dN_x : T_x M \to T_{N(x)} S^2$

is zero.

Lem Suppose that x lies in 3 lines in $Z(P)$. Then x is either a critical or a flat pt of $Z(P)$.

Pf Case 1 3 lines non-coplanar.

$\nabla P(x) \cdot u = 0$ in each of three directions

u corresponding to the 3 lines.

$\Rightarrow \nabla P(x) = 0$. Critical.

Case 2 $\nabla P(x) \neq 0$ x is a regular pt.

Translate & rotate so that $x = 0$.

and $Z(P)$ is given by $x_3 = h(x_1, x_2)$ locally at the origin, and the tangent plane of $Z(P)$ is $x_3 = 0$

Taylor expand: $h = h_2 + O(|x|^3)$

h_2 : homog poly of deg 2.

h_2 vanishes on 3 lines in the $x_1 x_2$-plane.

$h_2 \equiv 0$ by vanishing lemma

$\Rightarrow x$ is a flat point.

Flatness as an algebraic condition.

Critical is an alg condition:

$x \in Z(P)$

x critical $\Leftrightarrow$ $\underbrace{\nabla P(x)}_{\text{a list of 3 poly deg } < D} = 0$

Suppose x is a non-critical pt of $Z(P)$.

Let $N = \dfrac{\nabla P}{|\nabla P|}$ is the unit normal to $Z(P)$

x flat $\Leftrightarrow$ $\partial_v N(x) = 0 \quad \forall v \in T_x Z(P)$

How to write this as a polynomial?

Observe:

(1) $\partial_v N = 0 \Leftrightarrow \partial_v \nabla P$ is parallel to ∇P

$\Leftrightarrow \partial_v \nabla P \times \nabla P = 0$

(2) $\{e_1 \times \nabla P, e_2 \times \nabla P, e_3 \times \nabla P\}$ is a spanning set for $T_x Z(P)$.

Define

$$SP(x) : \left\{ \partial_{e_j \times \nabla P} \nabla P \times \nabla P \right\}_{j=1,2,3.}$$

$$\deg SP_j(x) \leq 3 \deg P$$

<u>Prop</u> $x \in Z(P)$.

$SP(x) = 0 \Leftrightarrow x$ is critical or flat

What happens when every pt of $Z(P)$ is flat?

<u>Lem</u> If P irreducible polynomial in $\mathbb{R}^3$, SP vanishes on $Z(P)$, and $Z(P)$ contains a regular pt, then $\deg P = 1$ and $Z(P)$ is a plane.

<u>Rmk</u> The assumption that $Z(P)$ has a reg pt is to prevent degeneracies e.g. $x_1^2 + x_2^2 = 0$

<u>Pf</u> Let $x \in Z(P)$ be the reg. pt, In neighborhood of x, $Z(P)$ is a flat submanifold, so its normal vector is constant, so its neighborhood is an open subset of a plane. By vanishing lemma, $Z(P)$ contains a whole plane. So P is divisible by a linear poly. Since P is irred, $\deg P = 1$ //

Plane detection lemma

$\forall P \in \mathrm{Poly}(\mathbb{R}^3)$. $\exists$ a list of nine poly. SP

(1) If $x \in Z(P)$, then
$SP(x) = 0 \iff x$ is critical or flat.

(2) If x is contained in 3 lines in $Z(P)$, then $SP(x) = 0$.

(3) $\deg SP \leq 3 \deg P$

(4) If P is irred, and SP vanishes on $Z(P)$, and $Z(P)$ contains a reg pt, then $Z(P)$ is a plane.

Proof of the planar clustering lemma.

$\mathcal{L}$: L lines in $\mathbb{R}^3$, each contains $\geq A \geq K L^{1/2}$ pts $\in P_3(\mathcal{L})$.

Let P be the min deg nonzero poly vanishing on $\mathcal{L}$.

By deg reduction, $\deg P \lesssim \frac{L}{A} \leq \frac{\sqrt{L}}{100}$

Factor P into irreducibles $P = \prod_j P_j$

Let $\mathcal{L}_{mult}$ be lines in $\mathcal{L}$ lying in multiple P_j's.

Bezout for lines:

$$|\mathcal{L}_{mult}| \leq \sum_{j,j'} (\deg P_j)(\deg P_{j'}) = (\deg P)^2 \leq \frac{1}{10^4} L$$

Let $\mathcal{L}_j$ be lines of $\mathcal{L}$ in $Z(P_j)$ but not other $Z(P_{j'})$'s.

<u>Lem</u> Each $l \in \mathcal{L}_j$ contains $\geq \frac{99}{100} A$ pts of $P_3(\mathcal{L}_j)$

<u>Pf</u> (Sketch). l contains $\geq A$ pts of $P_3(\mathcal{L})$. Few intersections with lines from other $Z(P_{j'})$ because l intersects $Z(P_{j'})$ at $\leq \deg P_{j'}$ pts. //

P min deg vanishing on $\mathcal{L}$

$\Rightarrow P_j$ is min deg poly vanishing on $\mathcal{L}_j$

$$\Rightarrow \deg P_j \lesssim \frac{|\mathcal{L}_j|}{A} \leq \frac{\sqrt{L_j}}{100}$$

At each $x \in P_3(\mathcal{L}_j)$,

$$SP_j(x) = 0$$

Use idea of contagious structure.

Each $l \in \mathcal{L}_j$ contains $\geq \frac{99}{100} A > 3 \deg P_j$ pts of $P_3(\mathcal{L}_j)$.

SP_j vanishes on $P_3(\mathcal{L}_j)$. $\Rightarrow SP_j$ vanishes on all $\mathcal{L}_j$

Since P_j is irred,

Bezout for lines

$\Rightarrow SP_j$ vanishes on $Z(P_j)$

or $Z(SP_j) \cap Z(P_j)$

has $\leq (\deg SP_j)(\deg P_j)$

$\leq 3 (\deg P_j)^2$

$< |L_j|$

$\therefore SP_j$ vanishes on $Z(P_j)$

It remains to show ~~that~~ $Z(P)$ contains a reg pt.

If not, ∇P_j vanishes on all lines, contradicts the min deg hypotheses.

$Z(P_j)$ is a plane.

$\#$ planes $\leq \deg P \lesssim \frac{L}{A}$

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