Quasirandom Cayley graphs

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(joint work with David Conlon)

A fundamental result of Chung, Graham, and Wilson [4], building on earlier work of Thomason [8, 9], states that for a sequence of graphs of constant edge-density, a number of seemingly distinct notions of quasirandomness are equivalent. In particular, for n-vertex, d-regular graphs, the following two properties are equivalent as long as $d = \Theta(n)$:

- *Discrepancy condition*: For all vertex subsets S and T,
$$e(S,T) = \frac{d}{n}|S||T| + o(nd);$$
- *Eigenvalue condition*: All eigenvalues of the the adjacency matrix, except the largest, are $o(d)$.

What about for sparse graphs, when $d = o(n)$?

The eigenvalue condition always implies the discrepancy condition. This is a consequence of the famous *expander mixing lemma*, which says that in an (n, d, λ)-*graph* (i.e., an n-vertex d-regular graph where all eigenvalues of the adjacency matrix, except the largest, are at most λ in absolute value), one has

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \le \lambda\sqrt{|S||T|} \tag{1}$$

for all vertex subsets S and T.

However, the discrepancy condition does not necessarily imply the eigenvalue condition when $d = o(n)$ [7, 3]. Consider the disjoint union of a large d-regular random graph and a copy of K_{d+1}. This graph satisfies the discrepancy condition since the copy of K_{d+1} does not significantly affect discrepancy. On the other hand, the eigenvalue d appears with multiplicity two (once for each connected component), so the graph does not satisfy the eigenvalue condition.

There have been some partial converses. For example, Bilu and Linial [2] gave a converse to the expander mixing lemma, showing that if (1) holds for all S and T, then the graph is an (n, d, λ')-graph with $\lambda' = O(\lambda \log d)$. The extra factor of $\log d$ cannot be removed. In a different direction, Alon et al. [1] showed that if the discrepancy condition is satisfied, then one can remove a $o(1)$-fraction of vertices from the graph so that remaining graph satisfies the eigenvalue condition.

A result of Kohayakawa, Rödl, and Schacht [6] (originally from 2003) comes as something of a surprise: the two properties are always equivalent for Cayley graphs of abelian groups. In our work [5], we extend their result to non-abelian groups, and more generally, all vertex-transitive graphs. Here is a precise statement of our theorem.

Theorem 1. *If an n-vertex d-regular Cayley graph (or more generally, a vertex-transitive graph) has the property that*

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \le \epsilon dn \tag{2}$$

for all vertex subsets S and T, then it is an (n,d,λ)-graph with $\lambda \le 8\epsilon d$.

The proof uses Grothendieck's inequality. We consider the cut norm for matrices, and show that its semidefinite relaxation equals the spectral norm when the matrix arises from a weighted Cayley graph. See our paper [5] for details.

References

[1] N. Alon, A. Coja-Oghlan, H. Hàn, M. Kang, V. Rödl and M. Schacht, Quasi-randomness and algorithmic regularity for graphs with general degree distributions, *SIAM J. Comput.* **39** (2010), 2336–2362.

[2] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap, *Combinatorica* **26** (2006), 495–519.

[3] B. Bollobás and V. Nikiforov, Hermitian matrices and graphs: singular values and discrepancy, *Discrete Math.* **285** (2004), 17–32.

[4] F. R. K. Chung, R. L. Graham, and R. M. Wilson, Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

[5] D. Conlon and Y. Zhao, Quasirandom Cayley graphs, *preprint*, available at arXiv:1603.03025 [math.CO].

[6] Y. Kohayakawa, V. Rödl, and M. Schacht, Discrepancy and eigenvalues of Cayley graphs, *preprint*, available at arXiv:1602.02291 [math.CO].

[7] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in More sets, graphs and numbers, Bolyai Soc. Math. Stud., Vol. 15, 199–262, Springer, Berlin, 2006.

[8] A. G. Thomason, Pseudorandom graphs, in Random graphs '85 (Poznań, 1985), North-Holland Math. Stud., Vol. 144, 307–331, North-Holland, Amsterdam, 1987.

[9] A. G. Thomason, Random graphs, strongly regular graphs and pseudorandom graphs, in Surveys in Combinatorics 1987, London Math. Soc. Lecture Note Ser., Vol. 123, 173–195, Cambridge University Press, Cambridge, 1987.