

**AN L^p THEORY OF SPARSE GRAPH CONVERGENCE I:
LIMITS, SPARSE RANDOM GRAPH MODELS,
AND POWER LAW DISTRIBUTIONS**

CHRISTIAN BORGS, JENNIFER T. CHAYES, HENRY COHN, AND YUFEI ZHAO

ABSTRACT. We introduce and develop a theory of limits for sequences of sparse graphs based on L^p graphons, which generalizes both the existing L^∞ theory of dense graph limits and its extension by Bollobás and Riordan to sparse graphs without dense spots. In doing so, we replace the no dense spots hypothesis with weaker assumptions, which allow us to analyze graphs with power law degree distributions. This gives the first broadly applicable limit theory for sparse graphs with unbounded average degrees. In this paper, we lay the foundations of the L^p theory of graphons, characterize convergence, and develop corresponding random graph models, while we prove the equivalence of several alternative metrics in a companion paper.

CONTENTS

1.	Introduction	1
2.	Definitions and results	4
3.	L^p graphons	15
4.	Regularity lemma for L^p upper regular graph(on)s	19
5.	Limit of an L^p upper regular sequence	25
6.	W -random weighted graphs	29
7.	Sparse random graphs	30
8.	Counting lemma for L^p graphons	33
	Acknowledgments	36
	Appendix A. L^p upper regularity implies unbounded average degree	36
	Appendix B. Proof of a Chernoff bound	37
	Appendix C. Uniform upper regularity	38
	References	42

1. INTRODUCTION

Understanding large networks is a fundamental problem in modern graph theory. What does it mean for two large graphs to be similar to each other, when they may differ in obvious ways such as their numbers of vertices? There are many types of networks (biological, economic, mathematical, physical, social, technological, etc.), whose details vary widely, but similar structural and growth phenomena occur in all these domains. In each case, it is natural to consider a sequence of graphs with

Zhao was supported by a Microsoft Research PhD Fellowship and internships at Microsoft Research New England.

size tending to infinity and ask whether these graphs converge to any meaningful sort of limit.

For dense graphs, the theory of graphons provides a comprehensive and flexible answer to this question (see, for example, [8, 9, 25, 26, 27]). Graphons characterize the limiting behavior of dense graph sequences, under several equivalent metrics that arise naturally in areas ranging from statistical physics to combinatorial optimization. Because dense graphs have been the focus of much of the graph theory developed in the last half century, graphons and related structural results about dense graphs play a foundational role in graph theory. However, many large networks of interest in other fields are sparse, and in the dense theory all sparse graph sequences converge to the zero graphon. This greatly limits the applicability of graphons to real-world networks. For example, in statistical physics dense graph sequences correspond to mean-field models, which are conceptually important as limiting cases but rarely applicable in real-world systems.

At the other extreme, there is a theory of graph limits for very sparse graphs, namely those with bounded degree or at least bounded average degree [1, 2, 4, 29] (see also [31, 32, 33, 34] for a broader framework based on first-order logic). Although this theory covers some important physical cases, such as crystals, it also does not apply to most networks of current interest. And although it is mathematically completely different in spirit from the theory of dense graph limits, it is also limited in scope. It covers the case of n -vertex graphs with $O(n)$ edges, while dense graph limits are nonzero only when there are $\Omega(n^2)$ edges.

Bollobás and Riordan [6] took an important step towards bridging the gap between these theories. They adapted the theory of graphons to sparse graphs by renormalizing to fix the effective edge density, which captures the intuition that two graphs with different densities may nevertheless be structurally similar. Under a boundedness assumption (Assumption 4.1 in [6]), which says that there are no especially dense spots within the graph, they showed that graphons remain the appropriate limiting objects. In other words, sparse graphs without dense spots converge to graphons after rescaling. Thus, these sparse graph sequences are characterized by their asymptotic densities and their limiting graphons.

The Bollobás-Riordan theory extends the scope of graphons to sparse graphs, but the boundedness assumption is nevertheless highly restrictive. In loose terms, it means the edge densities in different parts of the graph are all on roughly the same scale. By contrast, many of the most exciting network models have statistics governed by power laws [11, 30]. Such models generally contain dense spots, and we therefore must broaden the theory of graphons to handle them.

One setting in which these difficulties arise in practice is statistical estimation of network structure. Each graphon has a corresponding random graph model converging to it, and it is natural to try to fit these models to an observed network and thus estimate the underlying graphon (see, for example, [5]). Using the Bollobás-Riordan theory, Wolfe and Olhede [37] developed an estimator and proved its consistency under certain regularity conditions. Their theorems provide valuable statistical tools, but the use of the Bollobás-Riordan theory limits the applicability of their approach to graphs without dense spots and thus excludes many important cases.

In this paper, we develop an L^p theory of graphons for all $p > 1$, in contrast with the L^∞ theory studied in previous papers.¹ The L^p theory provides for the first time the flexibility to account for power laws, and we believe it is the right convergence theory for sparse graphs (outside of the bounded average degree regime). It generalizes dense graph limits and the Bollobás-Riordan theory, which together are the special case $p = \infty$, and it extends all the way to the natural barrier of $p = 1$.

It is also worth noting that, in the process of developing an L^p theory of graphons, we give a new L^p version of the Szemerédi regularity lemma for all $p > 1$ in its so-called weak (integral) form, which also naturally suggests the correct formulation for stronger forms. Long predating the theory of graph limits and graphons, it was recognized that the regularity lemma is a cornerstone of modern graph theory and indeed other aspects of discrete mathematics, so attempts were made to extend it to non-dense graphs. Our L^p version of the weak Szemerédi regularity lemma generalizes and extends previous work, as discussed below.

We will give precise definitions and theorem statements in §2, but first we sketch some examples motivating our theory.

We begin with dense graphs and L^∞ graphons. The most basic random graph model is the Erdős-Rényi model $G_{n,p}$, with n vertices and edges chosen independently with probability p between each pair of vertices. One natural generalization replaces p with a symmetric $k \times k$ matrix; then there are k blocks of n/k vertices each, with edge density $p_{i,j}$ between the i -th and j -th blocks. As $k \rightarrow \infty$, the matrix becomes a symmetric, measurable function $W: [0, 1]^2 \rightarrow [0, 1]$ in the continuum limit. Such a function W is an L^∞ graphon. All large graphs can be approximated by $k \times k$ block models with k large via Szemerédi regularity, from which it follows that limits of dense graph sequences are L^∞ graphons.

For sparse graphs the edge densities will converge to zero, but we would like a more informative answer than just $W = 0$. To determine the asymptotics, we rescale the density matrix p by a function of n so that it no longer tends to zero. In the Bollobás-Riordan theory, the boundedness assumption ensures that the densities are of comparable size (when smoothed out by local averaging) and hence remain bounded after rescaling. They then converge to an L^∞ graphon, and the known results on L^∞ graphons apply modulo rescaling.

For an example that cannot be handled using L^∞ graphons, consider the following configuration model. There are n vertices numbered 1 through n , with probability $\min(1, n^\beta(ij)^{-\alpha})$ of an edge between i and j , where $0 < \alpha < 1$ and $0 \leq \beta < 2\alpha$. In other words, the probabilities behave like $(ij)^{-\alpha}$, but boosted by a factor of n^β in case they become too small.² This model is one of the simplest ways to get a power law degree distribution, because the expected degree of vertex i scales according to an inverse power law in i with exponent α . The expected number of edges is on the order of $n^{\beta-2\alpha+2}$, which is superlinear when $\beta > 2\alpha - 1$. However, rescaling by the edge density $n^{\beta-2\alpha}$ does not yield an L^∞ graphon. Instead, we get $W(x, y) = (xy)^{-\alpha}$, which is unbounded.

¹The paper [24] and the online notes to Section 17.2 of [25] go a little beyond L^∞ graphons to study graphons in $\bigcap_{1 \leq p < \infty} L^p$.

²The inequalities $\alpha < 1$ and $\beta < 2\alpha$ each have a natural interpretation: the first avoids having almost all the edges between a sublinear number of vertices, while the second ensures that the cut-off from taking the minimum with 1 affects only a negligible fraction of the edges.

Unbounded graphons are of course far more expressive than bounded graphons, because they can handle an unbounded range of densities simultaneously. This issue does not arise for dense graphs: without rescaling, all densities are automatically bounded by 1. However, unboundedness is ubiquitous for sequences of sparse graphs.

To deal with unbounded graphons, we must reexamine the foundations of the theory of graphons. To have a notion of density at all, a graphon must at least be in $L^1([0,1]^2)$. Neglecting for the moment the limiting case of L^1 graphons, we show that L^p graphons are well behaved when $p > 1$. In the example above, the $p > 1$ case covers the full range $0 < \alpha < 1$, and we think of it as the primary case, while $p = 1$ is slightly degenerate and requires additional uniformity hypotheses (see Appendix C).

Each graphon W can be viewed as the archetype for a whole class of graphs, namely those that approximate it. It is natural to call these graphs W -quasirandom, because they behave as if they were randomly generated using W . From this perspective, the L^p theory of graphons completes the L^∞ theory: it adds the missing graphons that describe sparse graphs but not dense graphs.

The remainder of this paper is devoted to three primary tasks:

- (1) We lay the foundations of the L^p theory of graphons.
- (2) We characterize the sparse graph sequences that converge to L^p graphons via the concept of L^p upper regularity, and we establish the theory of convergence under the cut metric.
- (3) For each L^1 graphon W , we develop sparse W -random graph models and show that they converge to W .

Our main theorems are Theorems 2.8 and 2.14, which deal with tasks 2 and 3, respectively. Theorem 2.8 says that every L^p upper regular sequence of graphs with $p > 1$ has a subsequence that converges to an L^p graphon, and Theorem 2.14 says that sparse W -random graphs converge to W with probability 1. We also prove a number of other results, which we state in Section 2. One topic we do not address here is “right convergence” (notions of convergence based on quotients or statistical physics models). We analyze right convergence in detail in the companion paper [7].

2. DEFINITIONS AND RESULTS

2.1. Notation. We consider weighted graphs, which include as a special case simple unweighted graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively.

In a weighted graph G , every vertex $i \in V$ is given a weight $\alpha_i = \alpha_i(G) > 0$, and every edge $ij \in E(G)$ (allowing loops with $i = j$) is given a weight $\beta_{ij} = \beta_{ij}(G) \in \mathbb{R}$. We set $\beta_{ij} = 0$ whenever $ij \notin E(G)$. For each subset $U \subseteq V$, we write

$$\alpha_U := \sum_{i \in U} \alpha_i \quad \text{and} \quad \alpha_G := \alpha_{V(G)}.$$

We say a sequence $(G_n)_{n \geq 0}$ of weighted graphs has *no dominant nodes* if

$$\lim_{n \rightarrow \infty} \frac{\max_{i \in V(G_n)} \alpha_i(G_n)}{\alpha_{G_n}} = 0.$$

A simple (unweighted) graph is one in which $\alpha_i = 1$ for all $i \in V$, $\beta_{ij} = 1$ whenever $ij \in E$, and $\beta_{ij} = 0$ whenever $ij \notin E$. A simple graph contains no loops or multiple edges.

For $c \in \mathbb{R}$, we write cG for the weighted graph obtained from G by multiplying all edge weights by c , while the vertex weights remain unchanged.

For $1 \leq p \leq \infty$ we define the L^p norms

$$\|G\|_p := \left(\sum_{i,j \in V(G)} \frac{\alpha_i \alpha_j}{\alpha_G^2} |\beta_{ij}|^p \right)^{1/p} \quad \text{when } 1 \leq p < \infty$$

and

$$\|G\|_\infty := \max_{i,j \in V(G)} |\beta_{ij}|.$$

The quantity $\|G\|_1$ can be viewed as the edge density when G is a simple graph. When considering sparse graphs, we usually normalize the edge weights by considering the weighted graph $G/\|G\|_1$, in order to compare graphs with different edge densities. (Of course this assumes $\|G\|_1 \neq 0$, but that rules out only graphs with no edges, and we will often let this restriction pass without comment.)

In the previous works [8, 9] on convergence of dense graph sequences, only graphs with uniformly bounded $\|G\|_\infty$ were considered. In this paper, we relax this assumption. As we will see, this relaxation is useful even for sparse simple graphs due to the normalization $G/\|G\|_1$.

Given that we are relaxing the uniform bound on $\|G\|_\infty$, one might think, given the title of this paper, that we impose a uniform bound on $\|G\|_p$. This is *not* what we do. A bound on $\|G\|_p$ is too restrictive: for a simple graph G , an upper bound on $\|G/\|G\|_1\|_p = \|G\|_1^{\frac{1}{p}-1}$ corresponds to a lower bound on $\|G\|_1$, which forces G to be dense. Instead, we impose an L^p bound on edge densities with respect to vertex set partitions. This is explained next.

2.2. L^p upper regular graphs. For any $S, T \subseteq V(G)$, define the edge density (or average edge weight, for weighted graphs) between S and T by

$$\rho_G(S, T) := \sum_{s \in S, t \in T} \frac{\alpha_s \alpha_t}{\alpha_S \alpha_T} \beta_{st}.$$

We introduce the following hypothesis. Roughly speaking, it says that for every partition of the vertices of G in which no part is too small, the weighted graph derived from averaging the edge weights with respect to the partition is bounded in L^p norm (after normalizing by the overall edge density of the graph).

Definition 2.1. A weighted graph G (with vertex weights α_i and edge weights β_{ij}) is said to be (C, η) -upper L^p regular if $\alpha_i \leq \eta \alpha_G$ for all $i \in V(G)$, and whenever $V_1 \cup \dots \cup V_m$ is a partition of $V(G)$ into disjoint vertex sets with $\alpha_{V_i} \geq \eta \alpha_G$ for each i , one has

$$(2.1) \quad \sum_{i,j=1}^m \frac{\alpha_{V_i} \alpha_{V_j}}{\alpha_G^2} \left| \frac{\rho_G(V_i, V_j)}{\|G\|_1} \right|^p \leq C^p.$$

Informally, a graph G is (C, η) -upper L^p regular if $G/\|G\|_1$ has L^p norm at most C after we average over any partition of the vertices into blocks of at least $\eta|V(G)|$ in size (and no vertex has weight greater than $\eta \alpha_G$). We allow $p = \infty$, in which case (2.1) must be modified in the usual way to

$$\max_{1 \leq i, j \leq m} \left| \frac{\rho_G(V_i, V_j)}{\|G\|_1} \right| \leq C.$$

Strictly speaking, we should move $\|G\|_1$ to the right side of this inequality and (2.1), to avoid possibly dividing by zero, but we feel writing it this way makes the connection with $G/\|G\|_1$ clearer.

We will use the terms *upper L^p regular* and *L^p upper regular* interchangeably. The former is used so that we do not end up writing (C, η) *L^p upper regular*, which looks a bit odd.

Note that the definition of L^p upper regularity is interesting only for $p > 1$, since (2.1) automatically holds when $p = 1$ and $C = 1$. See Appendix C for a more refined definition, which plays the same role when $p = 1$.

Previous works on regularity and graph limits for sparse graphs (e.g., [6, 22]) assume a strong hypothesis, namely that $|\rho_G(S, T)| \leq C\|G\|_1$ whenever $|S|, |T| \geq \eta|V(G)|$. This is equivalent to what we call (C, η) -upper L^∞ regularity, and it is strictly stronger than L^p upper regularity for $p < \infty$. The relationship between these notions will become clearer when we discuss the graph limits in a moment. For now, it suffices to say that the limit of a sequence of L^p upper regular graphs is a graphon with a finite L^p norm.

2.3. Graphons. In this paper, we define the term *graphon* as follows.

Definition 2.2. A *graphon* is a symmetric, integrable function $W: [0, 1]^2 \rightarrow \mathbb{R}$.

Here *symmetric* means $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$. We will use λ to denote Lebesgue measure throughout this paper (on $[0, 1]$, $[0, 1]^2$, or elsewhere), and *measurable* will mean Borel measurable.

Note that in other books and papers, such as [8, 9, 25], the word “graphon” sometimes requires the image of W to be in $[0, 1]$, and the term *kernel* is then used to describe more general functions.

We define the L^p norm on graphons for $1 \leq p < \infty$ by

$$\|W\|_p := (\mathbb{E}[|W|^p])^{1/p} = \left(\int_{[0,1]^2} |W(x, y)|^p dx dy \right)^{1/p},$$

and $\|W\|_\infty$ is the essential supremum of W .

Definition 2.3. An *L^p graphon* is a graphon W with $\|W\|_p < \infty$.

By nesting of norms, an L^q graphon is automatically an L^p graphon for $1 \leq p \leq q \leq \infty$. Note that as part of the definition, we assumed all graphons are L^1 .

We define the inner product for graphons by

$$\langle U, W \rangle = \mathbb{E}[UW] = \int_{[0,1]^2} U(x, y)W(x, y) dx dy.$$

Hölder’s inequality will be very useful:

$$|\langle U, W \rangle| \leq \|U\|_p \|W\|_{p'},$$

where $1/p + 1/p' = 1$ and $1 \leq p, p' \leq \infty$. The special case $p = p' = 2$ is the Cauchy-Schwarz inequality.

Every weighted graph G has an associated graphon W^G constructed as follows. First divide the interval $[0, 1]$ into intervals $I_1, \dots, I_{|V(G)|}$ of lengths $\lambda(I_i) = \alpha_i/\alpha_G$ for each $i \in V(G)$. The function W^G is then given the constant value β_{ij} on $I_i \times I_j$ for all $i, j \in V(G)$. Note that $\|W^G\|_p = \|G\|_p$ for $1 \leq p \leq \infty$.

In the theory of dense graph limits, one proceeds by analyzing the associated graphons W^{G_n} for a sequence of graphs G_n , and in particular one is interested in the limit of W^{G_n} under the cut metric. However, for sparse graphs, where the density of the graphs tend to zero, the sequence W^{G_n} converges to an uninteresting limit of zero. In order to have a more interesting theory of sparse graph limits, we consider the normalized associated graphons $W^G / \|G\|_1$ instead.

Definition 2.4 (Stepping operator). For a graphon $W : [0, 1]^2 \rightarrow \mathbb{R}$ and a partition $\mathcal{P} = \{J_1, \dots, J_m\}$ of $[0, 1]$ into measurable subsets, we define a step-function $W_{\mathcal{P}} : [0, 1]^2 \rightarrow \mathbb{R}$ by

$$W_{\mathcal{P}}(x, y) := \frac{1}{\lambda(J_i)\lambda(J_j)} \int_{J_i \times J_j} W d\lambda \quad \text{for all } (x, y) \in J_i \times J_j.$$

In other words, $W_{\mathcal{P}}$ is produced from W by averaging over each cell $J_i \times J_j$.

A simple yet useful property of the stepping operator is that it is contractive with respect to the cut norm $\|\cdot\|_{\square}$ (defined in the next subsection) and all L^p norms, i.e., $\|W_{\mathcal{P}}\|_{\square} \leq \|W\|_{\square}$ and $\|W_{\mathcal{P}}\|_p \leq \|W\|_p$ for all graphon W and $1 \leq p \leq \infty$.

We can rephrase the definition of a (C, η) -upper L^p regular graph using the language of graphons. Let $V_1 \cup \dots \cup V_m$ be a partition of $V(G)$ as in Definition 2.1, and let $\mathcal{P} = \{J_1, \dots, J_m\}$, where J_i is the subset of $[0, 1]$ corresponding to V_i , i.e., $J_i = \bigcup_{v \in V_i} I_v$, where I_v is as in the definition of W^G . Then (2.1) simply says that

$$\|(W^G)_{\mathcal{P}}\|_p \leq C \|G\|_1.$$

This motivates the following notation of L^p upper regularity for graphons.

Definition 2.5. We say that a graphon $W : [0, 1]^2 \rightarrow \mathbb{R}$ is (C, η) -upper L^p regular if whenever \mathcal{P} is a partition of $[0, 1]$ into measurable sets each having measure at least η ,

$$\|W_{\mathcal{P}}\|_p \leq C.$$

Given a weighted graph G , if the normalized associated graphon $W^G / \|G\|_1$ is (C, η) -upper L^p regular and the vertex weights are all at most $\eta \alpha_G$, then G must also be (C, η) -upper L^p regular. The converse is not true, as the definition of upper regularity for graphons involves partitions \mathcal{P} of $[0, 1]$ that do not necessarily respect the vertex-atomicity of $V(G)$. For example, K_3 is a $(C, 1/2)$ -upper L^p regular graph for every $C > 0$ and $p > 1$ because no valid partition of vertices exist, but the same is not true for the graphon $W^{K_3} / \|K_3\|_1$.

2.4. Cut metric. The most important metric on the space of graphons is the cut metric. (Strictly speaking, it is merely a pseudometric, since two graphons with cut distance zero between them need not be equal.) It is defined in terms of the cut norm introduced by Frieze and Kannan [18].

Definition 2.6 (Cut metric). For a graphon $W : [0, 1]^2 \rightarrow \mathbb{R}$, define the *cut norm* by

$$(2.2) \quad \|W\|_{\square} := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) dx dy \right|,$$

where S and T range over measurable subsets of $[0, 1]$. Given two graphons $W, W' : [0, 1]^2 \rightarrow \mathbb{R}$, define

$$d_{\square}(W, W') := \|W - W'\|_{\square}$$

and the *cut metric* (or *cut distance*) δ_{\square} by

$$\delta_{\square}(W, W') := \inf_{\sigma} d_{\square}(W^{\sigma}, W'),$$

where σ ranges over all measure-preserving bijections $[0, 1] \rightarrow [0, 1]$ and $W^{\sigma}(x, y) := W(\sigma(x), \sigma(y))$.

For a survey covering many properties of the cut metric, see [21]. One convenient reformulation is that it is equivalent to the $L^{\infty} \rightarrow L^1$ operator norm, which is defined by

$$\|W\|_{\infty \rightarrow 1} = \sup_{\|f\|_{\infty}, \|g\|_{\infty} \leq 1} \left| \int_{[0,1]^2} W(x, y) f(x) g(y) dx dy \right|,$$

where f and g are functions from $[0, 1]$ to \mathbb{R} . Specifically, it is not hard to show that

$$(2.3) \quad \|W\|_{\square} \leq \|W\|_{\infty \rightarrow 1} \leq 4 \|W\|_{\square},$$

by checking that f and g take on only the values ± 1 in the extreme case.

We can extend the d and δ notations to any norm on the space of graphons. In particular, for $1 \leq p \leq \infty$, we define

$$d_p(W, W') := \|W - W'\|_p \quad \text{and} \quad \delta_p(W, W') := \inf_{\sigma} d_p(W^{\sigma}, W'),$$

with σ ranging overall measure-preserving bijections $[0, 1] \rightarrow [0, 1]$ as before.

To define the cut distance between two weighted graphs G and G' , we use their associated graphons. If G and G' are weighted graphs on the same set of vertices (with the same vertex weights), with edge weights given by $\beta_{ij}(G)$ and $\beta_{ij}(G')$ respectively, then we define

$$d_{\square}(G, G') := d_{\square}(W^G, W^{G'}) = \max_{S, T \subseteq V(G)} \left| \sum_{i \in S, j \in T} \frac{\alpha_i \alpha_j}{\alpha_G^2} (\beta_{st}(G) - \beta_{st}(G')) \right|,$$

where W^G and $W^{G'}$ are constructed using the same partition of $[0, 1]$ based on the vertex set. The final equality uses the fact that the cut norm for a graphon associated to a weighted graph can always be achieved by S and T in (2.2) that correspond to vertex subsets. This is due to the bilinearity of the expression of inside the absolute value in (2.2) with respect to the fractional contribution of each vertex to the sets S and T .

When G and G' have different vertex sets, $d_{\square}(G, G')$ no longer makes sense, but it still makes sense to define

$$\delta_{\square}(G, G') := \delta_{\square}(W^G, W^{G'}).$$

Similarly, for a weighted graph G and a graphon W , define

$$\delta_{\square}(G, W) := \delta_{\square}(W^G, W).$$

To compare graphs of different densities, we can compare the normalized associated graphons, i.e., $\delta_{\square}(G/\|G\|_1, G'/\|G'\|_1)$. We will sometimes refer to this quantity as the *normalized cut metric*.

2.5. L^p upper regular sequences.

Definition 2.7. Let $1 < p \leq \infty$ and $C > 0$. We say that $(G_n)_{n \geq 0}$ is a C -upper L^p regular sequence of weighted graphs if for every $\eta > 0$ there is some $n_0 = n_0(\eta)$ such that G_n is $(C + \eta, \eta)$ -upper L^p regular for all $n \geq n_0$. In other words, G_n is $(C + o(1), o(1))$ -upper L^p regular as $n \rightarrow \infty$. An L^p upper regular sequence of graphons is defined similarly.

As an example what kind of graphs this definition excludes, a sequence of graphs G_n formed by taking a clique on a subset of $o(|V(G_n)|)$ vertices and no other edges is not C -upper L^p regular for any $1 < p \leq \infty$ and $C > 0$. Furthermore, in Appendix A we show that the average degree in a C -upper L^p regular sequence of simple graphs must tend to infinity.

Now we are ready to state one of the main results of the paper, which asserts the existence of limits for L^p upper regular sequences.

Theorem 2.8. *Let $p > 1$ and let $(G_n)_{n \geq 0}$ be a C -upper L^p regular sequence of weighted graphs. Then there exists an L^p graphon W with $\|W\|_p \leq C$ so that*

$$\liminf_{n \rightarrow \infty} \delta_{\square} \left(\frac{G_n}{\|G_n\|_1}, W \right) = 0.$$

In other words, some subsequence of $G_n/\|G_n\|_1$ converges to W in the cut metric. An analogous result holds for L^p upper regular sequences of graphons.

Theorem 2.9. *Let $p > 1$ and let $(W_n)_{n \geq 0}$ be a C -upper L^p regular sequence of graphons. Then there exists an L^p graphon W with $\|W\|_p \leq C$ so that*

$$\liminf_{n \rightarrow \infty} \delta_{\square}(W_n, W) = 0.$$

These theorems, and all the remaining results in this subsection, are proved in §5.

The next proposition says that, conversely, every sequence that converges to an L^p graphon must be an L^p upper regular sequence.

Proposition 2.10. *Let $1 \leq p \leq \infty$, let W be an L^p graphon, and let $(W_n)_{n \geq 0}$ be a sequence of graphons with $\delta_{\square}(W_n, W) \rightarrow 0$ as $n \rightarrow \infty$. Then $(W_n)_{n \geq 0}$ is a $\|W\|_p$ -upper L^p regular sequence.*

An analogous result about weighted graphs follows as an immediate corollary by setting $W_n = W^{G_n}/\|G_n\|_1$.

Corollary 2.11. *Let $1 \leq p \leq \infty$, let W be an L^p graphon, and let $(G_n)_{n \geq 0}$ be a sequence of weighted graphs with no dominant nodes and with $\delta_{\square}(G_n/\|G_n\|_1, W) \rightarrow 0$ as $n \rightarrow \infty$. Then $(G_n)_{n \geq 0}$ is a $\|W\|_p$ -upper L^p regular sequence.*

The two limit results, Theorems 2.8 and 2.9, are proved by first developing a regularity lemma showing that one can approximate an L^p upper regular graph(on) by an L^p graphon with respect to cut metric, and then establishing a limit result in the space of L^p graphons. The latter step can be rephrased as a compactness result for L^p graphons, which we state in the next subsection.

We note that a sequence of graphs might not have a limit without the L^p upper regularity assumption. It could go wrong in two ways: (a) a sequence might not have any Cauchy subsequence, and (b) even a Cauchy sequence is not guaranteed to converge to a limit.

Proposition 2.12. (a) *There exists a sequence of simple graphs G_n so that*

$$\delta_{\square}(G_n/\|G_n\|_1, G_m/\|G_m\|_1) \geq 1/2 \quad \text{for all } n \text{ and } m \text{ with } n \neq m.$$

(b) *There exists a sequence of simple graphs G_n such that $(G_n/\|G_n\|_1)_{n \geq 0}$ is a Cauchy sequence with respect to δ_{\square} but does not converge to any graphon W with respect to δ_{\square} .*

2.6. Compactness of L^p graphons. Lovász and Szegedy [27] proved that the space of $[0, 1]$ -valued graphons is compact with respect to the cut distance (after identifying graphons with cut distance zero). We extend this result to L^p graphons.

Theorem 2.13 (Compactness of the L^p ball with respect to cut metric). *Let $1 < p \leq \infty$ and $C > 0$, and let $(W_n)_{n \geq 0}$ be a sequence of L^p graphons with $\|W_n\|_p \leq C$ for all n . Then there exists an L^p graphon W with $\|W\|_p \leq C$ so that*

$$\liminf_{n \rightarrow \infty} \delta_{\square}(W_n, W) = 0.$$

In other words, $\mathcal{B}_{L^p}(C) := \{L^p \text{ graphons } W : \|W\|_p \leq C\}$ is compact with respect to the cut metric δ_{\square} (after identifying points of distance zero).

For a proof, see §3. The analogous claim for $p = 1$ is false without additional hypotheses, as Proposition 2.12 implies that the L^1 ball of graphons is neither totally bounded nor complete with respect to δ_{\square} . The example showing that the L^1 ball is not totally bounded is easy: the sequence $W_n = 2^{2n} 1_{[2^{-n}, 2^{-n}] \times [2^{-n}, 2^{-n}]}$ satisfies $\delta_{\square}(W_n, W_m) > 1/2$ for every $m \neq n$. Our example showing incompleteness is a bit more involved, and we defer it to the proof of Proposition 2.12(b). See Theorem C.7 for an L^1 version of Theorem 2.13 under the hypothesis of uniform integrability.

2.7. Sparse W -random graph models. Our main result on this topic is that every graphon W gives rise to a natural random graph model, which produces a sequence of sparse graphs converging to W in the normalized cut metric. When W is nonnegative, the model produces sparse simple graphs. If W is allowed negative values, the resulting random graphs have ± 1 edge weights.

We explain this construction in two steps.

Step 1: From W to a random weighted graph. Given any graphon W , define $\mathbf{H}(n, W)$ to be a random weighted graph on n vertices (labeled by $[n] = \{1, 2, \dots, n\}$, with all vertex weights 1) constructed as follows: let x_1, \dots, x_n be i.i.d. chosen uniformly in $[0, 1]$, and then assign the weight of the edge ij to be $W(x_i, x_j)$ for all distinct $i, j \in [n]$.

Step 2: From a weighted graph to a sparse random graph. Let H be a weighted graph with $V(H) = [n]$ (with all vertex weights 1) and edge weights β_{ij} (with $\beta_{ii} = 0$), and let $\rho > 0$. When $\beta_{ij} \geq 0$ for all ij , the sparse random simple graph $\mathbf{G}(H, \rho)$ is defined by taking $V(H)$ to be the set of vertices and letting ij be an edge with probability $\min\{\rho\beta_{ij}, 1\}$, independently for all $ij \in E(H)$. If we allow negative edge weights on H , then we take $\mathbf{G}(H, \rho)$ to be a random graph with edge weights ± 1 , where ij is made an edge with probability $\min\{\rho|\beta_{ij}|, 1\}$ and given edge weight $+1$ if $\beta_{ij} > 0$ and -1 if $\beta_{ij} < 0$.

Finally, given any graphon W we define the sparse W -random (weighted) graph to be $\mathbf{G}(n, W, \rho) := \mathbf{G}(\mathbf{H}(n, W), \rho)$.

We also view $\mathbf{H}(n, W)$ and $\mathbf{G}(n, W, \rho_n)$ as graphons in the usual way, where the vertices are ordered according to the ordering of x_1, \dots, x_n as real numbers and

each vertex is represented by an interval of length $1/n$. For example, we use this interpretation in the notation $d_1(\mathbf{H}(n, W), W)$.

Note that it is also possible to consider other random weighted graph models where the edge weights are chosen from some other distribution (other than ± 1). Many of our results generalize easily, but we stick to our model for simplicity.

Here is our main theorem on W -random graphs. Note that we use the same i.i.d. sequence x_1, x_2, \dots for constructing $\mathbf{H}(n, W)$ and $\mathbf{G}(n, W, \rho_n)$ for different values of n , i.e., without resampling the x_i 's.

Theorem 2.14 (Convergence of W -random graphs). *Let W be an L^1 graphon.*

- (a) *We have $d_1(\mathbf{H}(n, W), W) \rightarrow 0$ as $n \rightarrow \infty$ with probability 1.*
- (b) *If $\rho_n > 0$ satisfies $\rho_n \rightarrow 0$ and $n\rho_n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$d_{\square}(\rho_n^{-1}\mathbf{G}(n, W, \rho_n), W) \rightarrow 0$$

as $n \rightarrow \infty$ with probability 1.

Part (a) is proved in §6 and part (b) in §7. Note that we use d_1 and d_{\square} (as opposed to δ_1 and δ_{\square}) because we have ordered the vertices of the graphs according to the ordering of the sample points x_1, \dots, x_n . Of course the sample point ordering is not determined by the graphs alone.

Corollary 2.15. *Let W be an L^1 graphon with $\|W\|_1 > 0$. Let $\rho_n > 0$ satisfy $\rho_n \rightarrow 0$ and $n\rho_n \rightarrow \infty$ as $n \rightarrow \infty$, and let $G_n = \mathbf{G}(n, W, \rho_n)$. Then*

$$\delta_{\square}(G_n / \|G_n\|_1, W / \|W\|_1) \rightarrow 0$$

as $n \rightarrow \infty$ with probability 1.

Furthermore, for any $1 \leq p \leq \infty$, if W is an L^p graphon, then $\|\mathbf{H}(n, W)\|_p \rightarrow \|W\|_p$ with probability 1 (this is an immediate consequence of Theorem 6.1 below). Thus $\mathbf{H}(n, W)$ generates a sequence of L^p graphons converging to W . Also, by Proposition 2.10 and Theorem 2.14(b), $\mathbf{G}(n, W, \rho_n)$ is a $\|W\|_p$ -upper L^p regular sequence that converges to W in normalized cut metric.

Note that the sparsity assumption $\rho_n \rightarrow 0$ is necessary since the edges of $\mathbf{G}(n, W, \rho_n)$ are included with probability $\min\{\rho_n |W(\cdot, \cdot)|, 1\}$, so ρ_n needs to be arbitrarily close to zero in order to “see” the unbounded part of W . Similarly, the assumption that $n\rho_n \rightarrow \infty$ means the expected average degree tends to infinity, which is necessary by Corollary 2.11 and Proposition A.1.

We will prove Theorem 2.14(a) using a theorem of Hoeffding on U -statistics, while Theorem 2.14(b) follows from Theorem 2.14(a) via a Chernoff-type argument that shows that if H is a weighted graph with many vertices, then $\rho^{-1}\mathbf{G}(H, \rho)$ is close to H in cut metric.

Theorem 2.14 was proved for L^∞ graphons as Theorem 4.5 in [8],³ but the proof given there does not seem to extend to Theorem 2.14. The proof here is much shorter than that in [8], though, unlike that proof, our proof gives no quantitative guarantees.

Using sparse W -random graphs, we can fully justify the name W -quasirandom for graphs approximating a graphon W . The following proposition shows that every sequence of sparse simple graphs converging to W is close in cut metric to W -random graphs:

³Technically, Theorem 4.5 in [8] is just a close analogue, since it uses δ_{\square} instead of d_1 and d_{\square} .

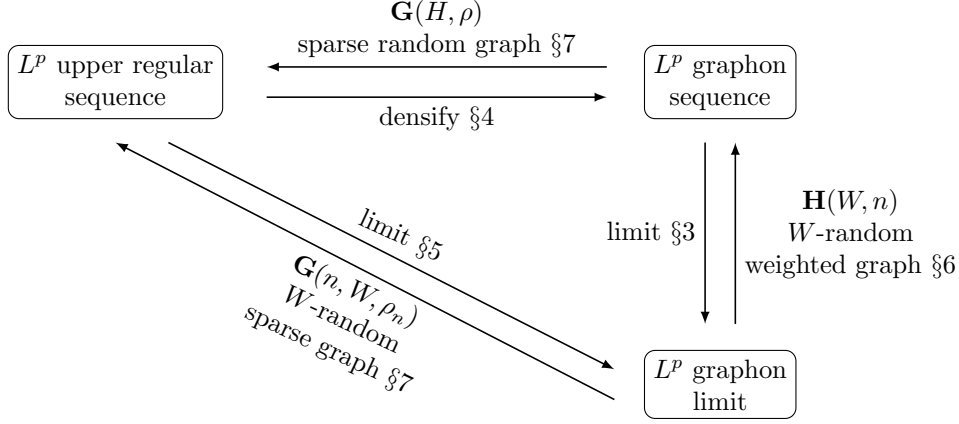


FIGURE 2.1. The relationships between the objects studied in this paper. The arrows are labeled with the relevant sections.

Proposition 2.16. *Let $p > 1$, and let $(G_n)_{n \geq 0}$ be a sequence of simple graphs such that $\|G_n\|_1 \rightarrow 0$ and $\delta_{\square}(G_n / \|G_n\|_1, W) \rightarrow 0$, where W is an L^p graphon. Let $G'_n = \mathbf{G}(|V(G_n)|, W, \|G_n\|_1)$. Then with probability 1, one can order the vertices of G_n and G'_n so that*

$$d_{\square} \left(\frac{G_n}{\|G_n\|_1}, \frac{G'_n}{\|G'_n\|_1} \right) \rightarrow 0.$$

See §7 for the proof, and Proposition C.16 for a generalization to $p = 1$.

2.8. From upper regular sequences to graphons and back. In Figure 2.1 we summarize the relationship between the objects studied in this paper. The inner set of arrows describe the process of going from a sequence to a limit, while the outer arrows describe the process of starting from a graphon W and constructing a sequence via a W -random graph model. Although we are primarily interested in the diagonal arrows connecting L^p upper regular sequences and L^p graphon limits, the proofs, in both directions, go through L^p graphons as a useful intermediate step.

We have not yet discussed the term *densify* in Figure 2.1. By densifying we mean approximating (in the sense of cut distance) an L^p upper regular graph by an L^p graphon. The former can be thought of as a sequence of sparse graphs with large edge weights supported on a sparse set of edges (although they do not have to be), and the latter as graphs on a dense set of edges with small weights (in the sense of being L^p bounded). More precisely, we prove the following result, which we think of as a transference theorem in the spirit of Green and Tao [19].

Proposition 2.17. *For every $p > 1$ and $\varepsilon > 0$ there exists an $\eta > 0$ such that for every (C, η) -upper L^p regular weighted graph G (or graphon W), there exists an L^p graphon U with $\|U\|_p \leq C$ such that*

$$\delta_{\square} \left(\frac{G}{\|G\|_1}, U \right) \leq C\varepsilon \quad (\text{respectively, } \delta_{\square}(W, U) \leq C\varepsilon).$$

We establish Proposition 2.17 as a weak regularity lemma. In fact, U can be constructed from G by averaging the edge weights over a partition of the vertex set

of G . As with other regularity lemmas, the number of parts used in the partition will be bounded. See §4 for the proof.

The regularity lemma for dense graphs was developed by Szemerédi [36]. Extensions of Szemerédi’s regularity lemma to sparse graphs were developed independently by Kohayakawa and Rödl [22, 23] under an L^∞ upper regularity assumption. Scott [35] gave another proof of a sparse regularity lemma without any assumptions, but as in Szemerédi’s regularity lemma, it allows for exceptional parts that could potentially hide all the “dense spots.” Frieze and Kannan [18] developed a weak version of regularity lemma with better bounds on the number of parts needed, and it is the version that we extend. This weak regularity lemma was extended to sparse graphs under the L^∞ upper regularity assumption in [6] and [12]. In our work, we extend the weak regularity lemma to L^p upper regular graphs.

Our proof of the weak regularity lemma for L^p upper regular graphs is an extension of the usual L^2 energy increment argument. However, the extension is not completely straightforward. Due to the nesting of norms, when $1 < p < 2$, we do not have very much control over the maximum L^2 energy for an L^p upper regular graph. This issue does not arise when $p \geq 2$ (e.g., $p = \infty$ in previous works). We resolve this issue via a careful truncation argument when $1 < p < 2$. As it turns out, these truncation arguments can be generalized to the case $p = 1$, provided we have sufficient control over the tails of W ; see Appendix C.

2.9. Counting lemma for L^p graphons. We have not yet addressed the issue of subgraph counts.⁴ For simple graphs F and G , a graph homomorphism from F to G is a map $V(F) \rightarrow V(G)$ that sends every edge of F to an edge of G . Let $\text{hom}(F, G)$ be the number of homomorphisms. The homomorphism density, or F -density, is defined by $t(F, G) := \text{hom}(F, G) / |V(G)|^{|V(F)|}$, which is equal to the probability that a random map $V(F) \rightarrow V(G)$ is a homomorphism.

In the theory of dense graph limits, the importance of homomorphism densities is that they characterize convergence under the cut metric: a sequence of dense graphs converges if and only if its F -densities converge for all F , and the limiting F -densities then describe the resulting graphon [8, Theorem 3.8]. This notion of convergence is called *left convergence*.

The situation is decidedly different for sparse graphs, and left convergence is not even implied by cut metric convergence, as we will see below. The irrelevance of left convergence is the most striking difference between dense and sparse graph limits, and it is an unavoidable consequence of sparsity. By contrast, right convergence (defined by quotients or statistical physics models) remains equivalent to metric convergence, as we show in [7].

⁴We actually only talk about homomorphism counts in this paper. There is a subtle yet significant distinction between homomorphisms and subgraphs, namely that subgraphs arise as homomorphisms for which the map $V(F) \rightarrow V(G)$ is injective. When G is a large, dense graph and F is fixed, this distinction is not important, since all but a vanishing proportion of maps $V(F) \rightarrow V(G)$ are injective. However, when G is sparse, this distinction could be significant (since the normalization is to divide the subgraph count by $\|G\|_1^{|E(F)|} |V(G)|^{|V(F)|}$). As an example, when $\rho = o(n^{-1/2})$, we have $n^4 \rho^4 = o(n^3 \rho^2)$, so the main contribution to the number of homomorphisms from C_4 to the random graph $G(n, \rho)$ is no longer coming from 4-cycles, but rather from paths of length 2 (each of which is the image of a homomorphism from C_4). However, as it turns out, we will not say much about either homomorphism densities or subgraph counts for sparse graphs anyway (our counting lemmas are for L^p graphons), so let us not dwell on the distinction between subgraphs and homomorphisms.

Before explaining further, we must extend the definition of homomorphism density to weighted graphs and graphons. For any simple graph F and graphon W , we define

$$t(F, W) := \int_{[0,1]^{|V(F)|}} \prod_{ij \in E(F)} W(x_i, x_j) dx_1 \dots dx_{|V(F)|}.$$

Note that $t(F, G) = t(F, W^G)$ for simple graphs G , and we take this as the definition of $t(F, G)$ for weighted graphs G .

A *counting lemma* is a claim that any two graphs/graphons that are close in cut metric must have similar F -densities. For dense graphs (or more generally, graphs with uniformly bounded edge weights), this claim is not too hard to show. For example, the following counting lemma appears in [8, Theorem 3.7(a)].

Theorem 2.18 (Counting lemma for L^∞ graphons). *Let F be a simple graph with m edges. If U and W are graphons with $\|U\|_\infty \leq 1$, $\|W\|_\infty \leq 1$, and $\delta_\square(U, W) \leq \varepsilon$, then*

$$|t(F, U) - t(F, W)| \leq 4m\varepsilon.$$

However, for sparse graphs, a general counting lemma of this form is too much to ask for, even for L^∞ upper regular graphs. Here is an example illustrating this difficulty. Let G_n be an instance of the Erdős-Rényi random graph $G(n, \rho_n)$, where $\rho_n > 0$ is the edge probability. If $n\rho_n \rightarrow \infty$, then $\rho_n^{-3}t(K_3, G_n) \rightarrow 1$ by a standard second moment argument, e.g., [3, Theorem 4.4.4]. Let G'_n be obtained from G_n by deleting edges from all triangles in G_n . If we additionally assume $\rho_n = o(n^{-1/2})$, so that $n^3\rho_n^3 = o(n^2\rho_n)$ and hence only an $o(1)$ fraction of the edges of G_n are deleted, then $d_\square(\rho_n^{-1}G_n, \rho_n^{-1}G'_n) = o(1)$. It follows that G_n and G'_n are close in (normalized) cut distance, but have very different (normalized) triangle densities, as $t(K_3, G'_n) = 0$. This example shows that we cannot expect a general counting lemma even for L^∞ upper regular sparse graphs, let alone L^p upper regular graphs.

Nevertheless, we will give a counting lemma for L^p graphons (which is the “dense setting,” as opposed to the “sparse setting” of L^p upper regular graphons). There is already an initial difficulty, which is that $t(F, W)$ might not be finite. The next proposition shows the conditions for $t(F, W)$ to be finite.

Proposition 2.19. *Let F be a simple graph with maximum degree Δ . For every $p < \Delta$, there exists an L^p graphon W with $t(F, W) = \infty$. On the other hand, if W is an L^Δ graphon, then $t(F, W)$ is well-defined and finite. Furthermore, $|t(F, W)| \leq \|W\|_\Delta^{|E(F)|}$.*

We want a counting lemma which asserts that if U and W are graphons with bounded L^p norms, then $|t(F, U) - t(F, W)|$ is small whenever $\delta_\square(U, W)$ is small. Proposition 2.19 suggests we should not expect such a counting lemma to hold when $p < \Delta$. In fact, we give a counting lemma whenever $p > \Delta$ and show that no counting lemma can hold when $p \leq \Delta$.

We prove the following extension of Theorem 2.18 to L^p graphons. Note that for fixed F and p , the bound in (2.4) is a function of ε that goes to zero as $\varepsilon \rightarrow 0$. As $p \rightarrow \infty$, the bound in Theorem 2.20 converges to that of Theorem 2.18.

Theorem 2.20 (Counting lemma for L^p graphons). *Let F be a simple graph with m edges and maximum degree Δ . Let $\Delta < p < \infty$. If U and W are graphons with*

$\|U\|_p \leq 1$, $\|W\|_p \leq 1$, and $\delta_{\square}(U, W) \leq \varepsilon$, then

$$(2.4) \quad |t(F, U) - t(F, W)| \leq 2m(m-1+p-\Delta) \left(\frac{2\varepsilon}{p-\Delta} \right)^{\frac{p-\Delta}{p-\Delta+m-1}}.$$

The counting lemma implies the following corollary for sequences of graphons that are uniformly bounded in L^p norm. As we saw above, L^p upper regularity would not suffice.

Corollary 2.21. *Let $p > 1$ and $C > 0$, and let W_n be a sequence of graphons converging to W in cut metric. Suppose $\|W_n\|_p \leq C$ for all n and $\|W\|_p \leq C$. Then for every simple graph F with maximum degree less than p , we have $t(F, W_n) \rightarrow t(F, W)$ as $n \rightarrow \infty$.*

On the other hand, no counting lemma can hold when $p \leq \Delta$, even if we replace the cut norm by the L^1 norm.

Proposition 2.22. *Let F be a simple graph with maximum degree $\Delta \geq 2$, and let $1 \leq p \leq \Delta$. Then there exists a sequence $(W_n)_{n \geq 0}$ of graphons with $\|W_n\|_p \leq 4$ such that $\|W_n - 1\|_1 \rightarrow 0$ as $n \rightarrow \infty$ yet*

$$\lim_{n \rightarrow \infty} t(F, W_n) = 2^{|\{v \in V(G) : \deg_F(v) = \Delta\}|} > 1 = t(F, 1).$$

See §8 for proofs of these results.

3. L^p GRAPHONS

Recall that an L^p graphon is a symmetric and integrable function $W : [0, 1]^2 \rightarrow \mathbb{R}$ with $\|W\|_p < \infty$. In this section, we prove Theorem 2.13, which gives a limit theorem for L^p graphons. The results in this section form the (L^p graphon sequence) \rightarrow (L^p graphon limit) arrow in Figure 2.1.

The proof technique is an extension of that of [27]. We will need a weak regularity lemma for L^p graphons. The standard proof of the weak regularity lemma involving L^2 energy increments, based on ideas from §8 of [18], works for L^2 graphons and hence L^p graphons for $p \geq 2$. Since several of our proofs are based on the same basic idea, we include the proof here. When $1 < p < 2$, we use a truncation argument to reduce to the $p = 2$ case.

Lemma 3.1 (Weak regularity lemma for L^2 graphons). *Let $\varepsilon > 0$, let $W : [0, 1]^2 \rightarrow \mathbb{R}$ be an L^2 graphon, and let \mathcal{P} be a partition of $[0, 1]$. Then there exists a partition \mathcal{Q} refining \mathcal{P} into at most $4^{1/\varepsilon^2} |\mathcal{P}|$ parts so that*

$$\|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon \|W\|_2.$$

Proof. We build a sequence $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots$ of partitions of $[0, 1]$, starting with $\mathcal{P}_0 = \mathcal{P}$. For each $i \geq 0$, the partition \mathcal{P}_{i+1} refines \mathcal{P}_i by dividing each part of \mathcal{P}_i into at most four subparts. So in particular $|\mathcal{P}_i| \leq 4^i |\mathcal{P}_0|$.

These partitions are constructed as follows. If for some i , \mathcal{P}_i satisfies $\|W - W_{\mathcal{P}_i}\|_{\square} \leq \varepsilon \|W\|_2$, then we stop. Otherwise, by the definition of the cut norm, there exists measurable subsets $S, T \subseteq [0, 1]$ with

$$|\langle W - W_{\mathcal{P}_i}, 1_{S \times T} \rangle| > \varepsilon \|W\|_2.$$

Let \mathcal{P}_{i+1} be the common refinement of \mathcal{P}_i with S and T . Since S and T are both unions of parts in \mathcal{P}_{i+1} ,

$$|\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, 1_{S \times T} \rangle| = |\langle W - W_{\mathcal{P}_i}, 1_{S \times T} \rangle| > \varepsilon \|W\|_2.$$

Since \mathcal{P}_{i+1} is a refinement of \mathcal{P}_i , $\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, W_{\mathcal{P}_i} \rangle = 0$. So by the Pythagorean theorem, followed by the Cauchy-Schwarz inequality,

$$\|W_{\mathcal{P}_{i+1}}\|_2^2 - \|W_{\mathcal{P}_i}\|_2^2 = \|W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}\|_2^2 \geq |\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, 1_{S \times T} \rangle|^2 > \varepsilon^2 \|W\|_2^2.$$

Since $\|W_{\mathcal{P}_i}\|_2 \leq \|W\|_2$ (by the convexity of $x \mapsto x^2$), we see that the process must stop with $i \leq 1/\varepsilon^2$. The final \mathcal{P}_i is the desired \mathcal{Q} . \square

An *equipartition* of $[0, 1]$ is a partition where all parts have equal measure. It will be convenient to enforce that the partitions obtained from the regularity lemma are equipartitions. The following lemma is similar to [25, Lemma 9.15(b)].

Lemma 3.2 (Equitizing a partition). *Let $p > 1$ and $\varepsilon > 0$, and let k be any positive integer. Let W be an L^p graphon, let \mathcal{P} be an equipartition of $[0, 1]$, and let \mathcal{Q} be a partition refining \mathcal{P} . Then there exists an equipartition \mathcal{Q}' refining \mathcal{P} into exactly $k|\mathcal{P}|$ parts so that*

$$\|W - W_{\mathcal{Q}'}\|_{\square} \leq 2 \|W - W_{\mathcal{Q}}\|_{\square} + 2 \|W\|_p \left(\frac{2|\mathcal{Q}|}{k|\mathcal{P}|} \right)^{1-1/p}.$$

Proof. For \mathcal{Q}' we choose any equipartition refining \mathcal{P} into exactly $k|\mathcal{P}|$ parts, at most $|\mathcal{Q}|$ of which intersect more than one part of \mathcal{Q} . We can construct such a \mathcal{Q}' as follows. For each part P_i of \mathcal{P} , let Q_{i1}, \dots, Q_{im} be the parts of \mathcal{Q} contained in P_i . Form \mathcal{Q}' by dividing up each of Q_{i1}, \dots, Q_{im} into parts of measure exactly $1/(k|\mathcal{P}|)$ plus a remainder part; then group the remainder parts in P_i together and divide them into parts of measure $1/(k|\mathcal{P}|)$. This partitions P_i into k parts of equal size. At most m of these new parts intersect more than one part of \mathcal{Q} , because there were at most m remainder parts, each of size less than $1/(k|\mathcal{P}|)$. Now carrying out this procedure for each part of \mathcal{P} gives an equipartition \mathcal{Q}' with the desired property.

Let \mathcal{R} be the common refinement of \mathcal{Q} and \mathcal{Q}' . Because the stepping operator is contractive with respect to the cut norm (i.e., $\|U_{\mathcal{R}}\|_{\square} \leq \|U\|_{\square}$),

$$\begin{aligned} \|W - W_{\mathcal{Q}'}\|_{\square} &\leq \|W - W_{\mathcal{Q}}\|_{\square} + \|W_{\mathcal{Q}} - W_{\mathcal{R}}\|_{\square} + \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_{\square} \\ &= \|W - W_{\mathcal{Q}}\|_{\square} + \|(W_{\mathcal{Q}} - W)_{\mathcal{R}}\|_{\square} + \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_{\square} \\ &\leq 2 \|W - W_{\mathcal{Q}}\|_{\square} + \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_{\square}. \end{aligned}$$

Thus, it will suffice to bound $\|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_{\square}$ by $2 \|W\|_p (2|\mathcal{Q}|/(k|\mathcal{P}|))^{1-1/p}$.

Let S be the union of the parts of \mathcal{Q}' that were broken up in its refinement \mathcal{R} . These are exactly the parts that intersect more than one part of \mathcal{Q} , so $\lambda(S) \leq |\mathcal{Q}|/(k|\mathcal{P}|)$. Using the agreement of $W_{\mathcal{Q}'}$ with $W_{\mathcal{R}}$ on $S^c \times S^c$ (where $S^c := [0, 1] \setminus S$), Hölder's inequality with $1/p + 1/p' = 1$, the bound $\|W_{\mathcal{R}}\|_p \leq \|W_{\mathcal{Q}'}\|_p \leq \|W\|_p$, and

the triangle inequality, we get

$$\begin{aligned}
\|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_{\square} &\leq \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_1 \\
&= \|(W_{\mathcal{R}} - W_{\mathcal{Q}'})(1 - 1_{S^c \times S^c})\|_1 \\
&\leq \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_p \|1 - 1_{S^c \times S^c}\|_{p'} \\
&= \|W_{\mathcal{R}} - W_{\mathcal{Q}'}\|_p (2\lambda(S) - \lambda(S)^2)^{1-1/p} \\
&\leq 2 \|W\|_p (2\lambda(S))^{1-1/p} \\
&\leq 2 \|W\|_p \left(\frac{2|\mathcal{Q}|}{k|\mathcal{P}|} \right)^{1-1/p},
\end{aligned}$$

as desired. \square

The following lemma is the L^2 version of Corollary 3.4(i) in [8], which in fact never required the L^∞ hypothesis implicitly assumed there.

Lemma 3.3 (Weak regularity lemma for L^2 graphons, equitable version). *Let $0 < \varepsilon < 1/3$ and let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be an L^2 graphon. Let \mathcal{P} be an equipartition of $[0, 1]$. Then for every integer $k \geq 4^{10/\varepsilon^2}$ there exists an equipartition \mathcal{Q} refining \mathcal{P} into exactly $k|\mathcal{P}|$ parts so that*

$$(3.1) \quad \|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon \|W\|_2.$$

Proof. Apply Lemma 3.1 to obtain a refinement \mathcal{Q} of \mathcal{P} into at most $4^{9/\varepsilon^2} |\mathcal{P}|$ parts so that $\|W - W_{\mathcal{Q}}\| \leq \frac{1}{3}\varepsilon \|W\|_2$. Now apply Lemma 3.2 with $p = 2$ to obtain a refinement \mathcal{Q}' of \mathcal{P} into an equipartition of exactly $k|\mathcal{P}|$ parts satisfying

$$\|W - W_{\mathcal{Q}'}\|_{\square} \leq 2 \|W - W_{\mathcal{Q}}\|_{\square} + 2 \|W\|_2 \sqrt{\frac{2|\mathcal{Q}|}{k|\mathcal{P}|}} \leq 2 \cdot \frac{\varepsilon}{3} \|W\|_2 + 2 \|W\|_2 \cdot \frac{\varepsilon}{6} \leq \varepsilon \|W\|_2.$$

Here we used $|\mathcal{Q}'|/|\mathcal{P}| \leq 4^{9/\varepsilon^2} \leq \frac{\varepsilon^2}{72} 4^{10/\varepsilon^2} \leq \frac{1}{2} \left(\frac{\varepsilon}{6}\right)^2 k$, which holds for $0 < \varepsilon < 1/3$. So \mathcal{Q}' is the desired partition. \square

Lemma 3.3 also works for L^p graphons for all $p \geq 2$ by nesting of norms, as (3.1) implies $\|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon \|W\|_p$. Now we deal with the case $1 < p < 2$.

Lemma 3.4 (Weak regularity lemma for L^p graphons). *Let $1 < p < 2$ and $0 < \varepsilon < 1$. Let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be an L^p graphon. Let \mathcal{P} be an equipartition of $[0, 1]$. Then for any integer $k \geq 4^{10(3/\varepsilon)^{p/(p-1)}}$ there exists an equipartition \mathcal{Q} refining \mathcal{P} into exactly $k|\mathcal{P}|$ parts so that*

$$\|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon \|W\|_p.$$

Note that as $p \nearrow 2$, the exponent $p/(p-1)$ of $1/\varepsilon$ in k in the lemma tends to 2, which is the best possible exponent in the bound for the weak regularity lemma when $p \geq 2$ by [13].

Proof. Set $K = (3/\varepsilon)^{1/(p-1)} \|W\|_p$, and let

$$W' = W 1_{|W| \leq K}.$$

We have

$$\begin{aligned}\|W'\|_2 &= \|W1_{|W|\leq K}\|_2 \\ &\leq \left\|W(K/|W|)^{1-p/2}\right\|_2 \\ &= \|W\|_p^{p/2} K^{1-p/2} = (3/\varepsilon)^{\frac{2-p}{2(p-1)}} \|W\|_p.\end{aligned}$$

By Lemma 3.3 there exists an equitable partition \mathcal{Q} refining \mathcal{P} into exactly $k|\mathcal{P}|$ parts so that

$$\|W' - W'_\mathcal{Q}\|_\square \leq \left(\frac{\varepsilon}{3}\right)^{\frac{p}{2(p-1)}} \|W'\|_2 \leq \frac{\varepsilon}{3} \|W\|_p.$$

We also have

$$\begin{aligned}\|W_\mathcal{Q} - W'_\mathcal{Q}\|_1 &= \|(W - W')_\mathcal{Q}\|_1 \\ &\leq \|W - W'\|_1 = \|W1_{|W|>K}\|_1 \\ &\leq \|W(|W|/K)^{p-1}\|_1 = \|W\|_p^p / K^{p-1} = \frac{\varepsilon}{3} \|W\|_p.\end{aligned}$$

It follows that

$$\begin{aligned}\|W - W_\mathcal{Q}\|_\square &\leq \|W - W'\|_\square + \|W' - W'_\mathcal{Q}\|_\square + \|W'_\mathcal{Q} - W_\mathcal{Q}\|_\square \\ &\leq \|W - W'\|_1 + \|W' - W'_\mathcal{Q}\|_\square + \|W'_\mathcal{Q} - W_\mathcal{Q}\|_1 \\ &\leq \frac{\varepsilon}{3} \|W\|_p + \frac{\varepsilon}{3} \|W\|_p + \frac{\varepsilon}{3} \|W\|_p = \varepsilon \|W\|_p.\end{aligned}$$

Therefore \mathcal{Q} is the desired partition. \square

Now we prove that the L^p ball is compact with respect to the cut metric.

Proof of Theorem 2.13. The proof of the theorem is a small modification of the argument in [27, Theorem 5.1], with adaptations to the L^p setting. We begin by using the weak regularity lemmas to produce approximations to the sequence $(W_n)_{n \geq 0}$. The approximations using a fixed number of parts are easier to analyze than the original sequence, because they involve only a finite amount of information. We take limits of these approximations and show that they form a martingale as one varies the number of parts. The limit of the original sequence is then derived using the martingale convergence theorem.

By scaling we may assume without loss of generality that $C = 1$. For each k and n we construct an equipartition $\mathcal{P}_{n,k}$ using Lemma 3.3 (when $p \geq 2$) or Lemma 3.4 (when $1 < p < 2$), so that

$$\|W_n - (W_n)_{\mathcal{P}_{n,k}}\|_\square \leq 1/k.$$

In doing so, we may assume that $\mathcal{P}_{n,k+1}$ always refines $\mathcal{P}_{n,k}$ and that $|\mathcal{P}_{n,k}|$ is independent of n .

The first step is to change variables so the partitions $\mathcal{P}_{n,k}$ become the same. Let \mathcal{P}_k be a partition of $[0, 1]$ into $|\mathcal{P}_{n,k}|$ intervals of equal length, and for each n and k , let $\sigma_{n,k}$ be a measure-preserving bijection from $[0, 1]$ to itself that transforms $\mathcal{P}_{n,k}$ into \mathcal{P}_k . (This can always be done; see, for example, Theorem A.7 in [21].) Now let

$$W_{n,k} = (W_n^{\sigma_{n,k}})_{\mathcal{P}_k} = ((W_n)_{\mathcal{P}_{n,k}})^{\sigma_{n,k}}.$$

Then $W_{n,k}$ is a step-function with interval steps formed from \mathcal{P}_k , and

$$\delta_\square(W_n, W_{n,k}) \leq 1/k.$$

Since each interval of \mathcal{P}_k has length exactly $1/|\mathcal{P}_k|$ and the stepping operator is contractive with respect to the p -norm,

$$|\mathcal{P}_k|^{-2} \|W_{n,k}\|_\infty^p \leq \|W_{n,k}\|_p^p \leq \|W_n\|_p^p \leq 1.$$

Thus $\|W_{n,k}\|_\infty \leq |\mathcal{P}_k|^{2/p}$.

We next pass to a subsequence of $(W_n)_{n \geq 0}$ such that for each k , $W_{n,k}$ converges to a limit U_k almost everywhere as $n \rightarrow \infty$. For each fixed k , this is easily done using compactness of a $|\mathcal{P}_k|^2$ -dimensional cube, because the function $W_{n,k}$ is determined by $|\mathcal{P}_k|^2$ values corresponding to pairs of parts in \mathcal{P}_k and $\|W_{n,k}\|_\infty$ is uniformly bounded. To find a single subsequence that ensures convergence for all k , we iteratively choose a subsequence for $k = 1, 2, \dots$.

For each k , the limit U_k is a step function with $|\mathcal{P}_k|$ steps such that $\|W_{n,k} - U_k\|_p \rightarrow 0$ as $n \rightarrow \infty$. In particular, this implies that $\|U_k\|_p \leq 1$ for all k , since $\|W_{n,k}\|_p \leq \|W_n\|_p \leq 1$ for all n and k .

The crucial property of the sequence U_1, U_2, \dots is that it forms a martingale on $[0, 1]^2$ with respect to the σ -algebras generated by the products of the parts of $\mathcal{P}_1, \mathcal{P}_2, \dots$. In other words, $(U_{k+1})_{\mathcal{P}_k} = U_k$. This follows immediately from

$$(W_{n,k+1})_{\mathcal{P}_k} = (W_n^{\sigma_{n,k+1}})_{\mathcal{P}_k} = ((W_n)_{\mathcal{P}_{n,k}})^{\sigma_{n,k+1}} = W_{n,k}.$$

(Note that $\sigma_{n,k+1}$ transforms $\mathcal{P}_{n,k}$ into \mathcal{P}_k because it does the same for their refinements $\mathcal{P}_{n,k+1}$ and \mathcal{P}_{k+1} .)

By the L^p martingale convergence theorem [16, Theorem 5.4.5], there exists some $W \in L^p([0, 1]^2)$ such that $\|U_k - W\|_p \rightarrow 0$ as $k \rightarrow \infty$. Since $\|U_k\|_p \leq 1$ for all k , we have $\|W\|_p \leq 1$.

Now W is the desired limit, because

$$\begin{aligned} \delta_\square(W_n, W) &\leq \delta_\square(W_n, W_{n,k}) + \delta_\square(W_{n,k}, U_k) + \delta_\square(U_k, W) \\ &\leq \delta_\square(W_n, W_{n,k}) + \|W_{n,k} - U_k\|_1 + \|U_k - W\|_1. \end{aligned}$$

Each of the terms in this bound can be made arbitrarily small by choosing k and then n large enough. Thus, $\delta_\square(W_n, W) \rightarrow 0$ as $n \rightarrow \infty$, as desired (keeping in mind that we have passed to a subsequence). \square

4. REGULARITY LEMMA FOR L^p UPPER REGULAR GRAPH(ON)S

In this section we prove a regularity lemma for L^p upper regular graphs and graphons. This forms the (L^p upper regular sequence) \rightarrow (L^p graphon sequence) arrow in Figure 2.1. We will first present the proof for graphons, since the notation is somewhat simpler. Then we will explain the minor modifications needed to prove the result for weighted graphs. The difference between the two settings is that for graphs, the partitions of $[0, 1]$ in the corresponding graphon need to respect the atomicity of the vertices, but this is only a minor inconvenience since the L^p upper regularity condition ensures that no vertex has weight too large.

The main ideas of the proof are as follows. Suppose W is a (C, η) -upper L^p regular graphon with $p \geq 2$. We would like to proceed as in the proof of the L^2 weak regularity lemma, by constructing partitions $\mathcal{P}_0, \mathcal{P}_1, \dots$ such that if $\|W - W_{\mathcal{P}_i}\|_\square > C\varepsilon$, then

$$\|W_{\mathcal{P}_{i+1}}\|_2^2 \geq \|W_{\mathcal{P}_i}\|_2^2 + (C\varepsilon)^2.$$

Furthermore, we would like all the parts of \mathcal{P}_i to have measure at least η , so that $\|W_{\mathcal{P}_i}\|_2 \leq \|W_{\mathcal{P}_i}\|_p \leq C$. These bounds cannot both hold for all i , so we must eventually have $\|W - W_{\mathcal{P}_i}\|_{\square} \leq C\varepsilon$ for some i .

When we try to do this, we run into two problems:

- (1) While $\|W - W_{\mathcal{P}_i}\|_{\square} > C\varepsilon$ gives sets S and T such that $|\langle W - W_{\mathcal{P}_i}, 1_{S \times T} \rangle| > C\varepsilon$, the partition generated by \mathcal{P}_i , S , and T may have a part of size less than η . In that case, we cannot use the upper regularity assumption as we proceed.
- (2) When $p < 2$, the L^2 increment argument does not work, since we only have bounds on $\|W_{\mathcal{P}_i}\|_p$, not $\|W_{\mathcal{P}_i}\|_2$.

To deal with the first problem, we will modify S and T to S' and T' such that the new partition has large enough parts, while $|\langle W - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| > C\varepsilon/2$. To do so, we will need a technical lemma, Lemma 4.2 below, which allows us to bound the difference between these inner products, and which itself follows from a simpler lemma, Lemma 4.1. After stating and proving these lemmas, we will formulate Theorem 4.3, which is the regularity lemma version of Proposition 2.17 for graphons. In its proof, we deal with the first problem as describe above, while we deal with the second by a suitable truncation argument.

We begin with a lemma that bounds the weight of W on $1_{S \times T}$ when one of S and T is small. Recall that λ denotes Lebesgue measure.

Lemma 4.1. *Assume $\eta < 1/9$. Let $W : [0, 1]^2 \rightarrow \mathbb{R}$ be a (C, η) -upper L^p regular graphon, and let $S, T \subseteq [0, 1]$ be measurable subsets. If $\lambda(S) \leq \delta$ for some $\delta \geq \eta$, then*

$$|\langle W, 1_{S \times T} \rangle| \leq 10C\delta^{1-1/p}.$$

Proof. We prove the lemma in three steps.

Step 1. Let \mathcal{P} be the smallest partition of $[0, 1]$ that simultaneously refines S and T (i.e., the parts are $S \cap T, S^c \cap T, S \cap T^c, S^c \cap T^c$, excluding empty parts, where $S^c := [0, 1] \setminus S$). If all parts of \mathcal{P} have measure at least η , then we can apply Hölder's inequality (with $1/p + 1/p' = 1$) and the (C, η) -upper L^p regularity hypothesis to conclude

$$|\langle W, 1_{S \times T} \rangle| = |\langle W_{\mathcal{P}}, 1_{S \times T} \rangle| \leq \|W_{\mathcal{P}}\|_p \|1_{S \times T}\|_{p'} \leq C(\lambda(S)\lambda(T))^{1-1/p}.$$

Step 2. In this step we assume that $3\eta \leq \lambda(T) \leq 1 - 3\eta$. The partition \mathcal{P} generated by S and T as in Step 1 might not satisfy the condition of all parts having measure at least η . Define $S_1 \subseteq T$ and $S_2 \subseteq T^c$ as follows.

If $\lambda(S \cap T) < \eta$, then let S_1 be an arbitrary subset of $T \setminus S$ with $\lambda(S_1) = \eta$; else, if $\lambda(S^c \cap T) < \eta$ (equivalently, $\lambda(S \cap T) > \lambda(T) - \eta$), then let S_1 be an arbitrary subset of $S \cap T$ with $\lambda(S_1) = \eta$; else, let $S_1 = \emptyset$.

Similarly, if $\lambda(S \cap T^c) < \eta$, then let S_2 be an arbitrary subset of $T^c \setminus S$ with $\lambda(S_2) = \eta$; else, if $\lambda(S \cap T^c) > \lambda(T^c) - \eta$, then let S_2 be an arbitrary subset of $S \cap T^c$ with $\lambda(S_2) = \eta$; else, let $S_2 = \emptyset$.

Let $S' = S \triangle S_1 \triangle S_2$ (where \triangle denotes the symmetric difference, and here each S_i is either contained in S or disjoint from S). Note that the pairs $(S_1, T), (S_2, T)$,

(S', T) all satisfy the hypotheses of Step 1. So we have

$$\begin{aligned}
|\langle W, 1_{S \times T} \rangle| &= |\langle W, 1_{S' \times T} \pm 1_{S_1 \times T} \pm 1_{S_2 \times T} \rangle| \\
&\leq |\langle W, 1_{S' \times T} \rangle| + |\langle W, 1_{S_1 \times T} \rangle| + |\langle W, 1_{S_2 \times T} \rangle| \\
&\leq C(\lambda(S')\lambda(T))^{1-1/p} + C(\lambda(S_1)\lambda(T))^{1-1/p} + C(\lambda(S_2)\lambda(T))^{1-1/p} \\
&\leq C(\lambda(S) + 2\eta)^{1-1/p} + 2C\eta^{1-1/p} \\
&\leq 5C\delta^{1-1/p}.
\end{aligned}$$

The last step follows from the assumption $\lambda(S) \leq \delta$ and $\delta \geq \eta$.

Step 3. Now we relax the $3\eta \leq \lambda(T) \leq 1 - 3\eta$ assumption. If $\lambda(T) < 3\eta$, then let T_1 be any subset of T^c with $\lambda(T_1) = 3\eta$; else, if $\lambda(T) > 1 - 3\eta$, then let T_1 be any subset of T with $\lambda(T_1) = 3\eta$; else, let $T_1 = \emptyset$. Let $T' = T \Delta T_1$. Then $3\eta \leq \lambda(T') \leq 1 - 3\eta$. So applying Step 2, we have

$$|\langle W, 1_{S \times T} \rangle| \leq |\langle W, 1_{S \times T'} \rangle| + |\langle W, 1_{S \times T_1} \rangle| \leq 10C\delta^{1-1/p}. \quad \square$$

Lemma 4.2. *Assume $\eta < 1/9$. Let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be a (C, η) -upper L^p regular graphon. Let $S, S', T, T' \subseteq [0, 1]$ be measurable sets satisfying $\lambda(S \Delta S'), \lambda(T \Delta T') \leq \delta$, for some $\delta \geq \eta$. Then*

$$|\langle W, 1_{S \times T} - 1_{S' \times T'} \rangle| \leq 40C\delta^{1-1/p}.$$

Proof. We have

$$1_{S \times T} - 1_{S' \times T'} = 1_{(S \setminus S') \times T} + 1_{(S \cap S') \times (T \setminus T')} - 1_{(S' \setminus S) \times T'} - 1_{(S \cap S') \times (T' \setminus T)}.$$

Applying Lemma 4.1 to each of the four terms below and using $\lambda(S \setminus S'), \lambda(S' \setminus S), \lambda(T \setminus T'), \lambda(T' \setminus T) \leq \delta$, we have

$$\begin{aligned}
|\langle W, 1_{S \times T} - 1_{S' \times T'} \rangle| &\leq |\langle W, 1_{(S \setminus S') \times T} \rangle| + |\langle W, 1_{(S \cap S') \times (T \setminus T')} \rangle| \\
&\quad + |\langle W, 1_{(S' \setminus S) \times T'} \rangle| + |\langle W, 1_{(S \cap S') \times (T' \setminus T)} \rangle| \\
&\leq 4 \cdot 10C\delta^{1-1/p}. \quad \square
\end{aligned}$$

Theorem 4.3 (Weak regularity lemma for L^p upper regular graphons). *Let $C > 0$, $p > 1$, and $0 < \varepsilon < 1$. Set $N = (6/\varepsilon)^{\max\{2, p/(p-1)\}}$ and $\eta = 4^{-N-1}(\varepsilon/160)^{p/(p-1)}$. Let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be a (C, η) -upper L^p regular graphon. Then there exists a partition \mathcal{P} of $[0, 1]$ into at most 4^N measurable parts, each having measure at least η , so that*

$$\|W - W_{\mathcal{P}}\|_{\square} \leq C\varepsilon.$$

Proposition 2.17 for graphons follows as an immediate corollary.

Proof. We consider a sequence of partitions $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ of $[0, 1]$, starting with the trivial partition $\mathcal{P}_0 = \{[0, 1]\}$. The following properties will be maintained:

- (1) The partition \mathcal{P}_{i+1} refines \mathcal{P}_i by dividing each part of \mathcal{P}_i into at most four subparts. So in particular $|\mathcal{P}_i| \leq 4^i$.
- (2) For each i , all parts of \mathcal{P}_i have measure at least η .

These partitions are constructed as follows. For each $0 \leq i < n$, if \mathcal{P}_i satisfies $\|W - W_{\mathcal{P}_i}\|_{\square} \leq C\varepsilon$, then we have found the desired partition. Otherwise, there exists measurable subsets $S, T \subseteq [0, 1]$ with

$$(4.1) \quad |\langle W - W_{\mathcal{P}_i}, 1_{S \times T} \rangle| > C\varepsilon.$$

Next we find S', T' so that $\lambda(S \Delta S'), \lambda(T \Delta T') \leq 2|\mathcal{P}_i|\eta$, such that if we define \mathcal{P}_{i+1} to be the common refinement of \mathcal{P}, S' , and T' , then all parts of \mathcal{P}_i have size at least η . Indeed, look at the intersection of S with each part of \mathcal{P}_i , and obtain S' from S by deleting (rounding down) the parts that intersect with S in measure less than η , and then adding (rounding up) the parts that intersect S^c in measure less than η . Let $\mathcal{P}_{i+1/2}$ be the common refinement of \mathcal{P}_i and S' , so that all parts of $\mathcal{P}_{i+1/2}$ have measure at least η , and $\lambda(S \Delta S') \leq |\mathcal{P}_i|\eta$. Next, do a similar procedure to T to obtain T' so that the common refinement \mathcal{P}_{i+1} of $\mathcal{P}_{i+1/2}$ and T' has all parts with measure at least η . Here we have $\lambda(T \Delta T') \leq |\mathcal{P}_{i+1/2}|\eta \leq 2|\mathcal{P}_i|\eta$. So \mathcal{P}_{i+1} has the desired properties.

If the construction of the sequence $\mathcal{P}_0, \dots, \mathcal{P}_n$ of partitions stops with $n \leq N$, then we are done. Otherwise let us stop the sequence at \mathcal{P}_n with $n = \lceil N \rceil$. We will derive a contradiction.

Let $0 \leq i < n$, and let S, S', T, T' be the sets used to construct \mathcal{P}_{i+1} from \mathcal{P}_i . Using $\lambda(S \Delta S'), \lambda(T \Delta T') \leq 2|\mathcal{P}_i|\eta \leq 2 \cdot 4^N \eta$, we have by Lemma 4.2

$$(4.2) \quad |\langle W, 1_{S \times T} - 1_{S' \times T'} \rangle| \leq 40C(2 \cdot 4^N \eta)^{1-1/p} \leq C\varepsilon/4.$$

Also by Hölder's inequality (with $1/p + 1/p' = 1$),

$$(4.3) \quad \begin{aligned} |\langle W_{\mathcal{P}_i}, 1_{S \times T} - 1_{S' \times T'} \rangle| &\leq \|W_{\mathcal{P}_i}\|_p \|1_{S \times T} - 1_{S' \times T'}\|_{p'} \\ &\leq C(\lambda(S \Delta S') + \lambda(T \Delta T'))^{1/p'} \\ &\leq C(4 \cdot 4^N \eta)^{1-1/p} \leq C\varepsilon/160 \leq C\varepsilon/8. \end{aligned}$$

It follows that

$$\begin{aligned} |\langle W - W_{\mathcal{P}_i}, 1_{S \times T} \rangle - \langle W - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| &\leq |\langle W, 1_{S \times T} - 1_{S' \times T'} \rangle| \\ &\quad + |\langle W_{\mathcal{P}_i}, 1_{S \times T} - 1_{S' \times T'} \rangle| \\ &\leq C\varepsilon/2. \end{aligned}$$

Combing the above inequality with (4.1) gives us

$$|\langle W - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| > C\varepsilon/2.$$

Since S' and T' are both unions of parts in \mathcal{P}_{i+1} , we have $\langle W, 1_{S' \times T'} \rangle = \langle W_{\mathcal{P}_{i+1}}, 1_{S' \times T'} \rangle$, so

$$(4.4) \quad |\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| > C\varepsilon/2.$$

We consider two cases: $p \geq 2$ and $1 < p < 2$.

Case I: $p \geq 2$. This case is easier. Since \mathcal{P}_{i+1} is a refinement of \mathcal{P}_i , we have $\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, W_{\mathcal{P}_i} \rangle = 0$. So by the Pythagorean theorem, followed by the Cauchy-Schwarz inequality,

$$\|W_{\mathcal{P}_{i+1}}\|_2^2 - \|W_{\mathcal{P}_i}\|_2^2 = \|W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}\|_2^2 \geq |\langle W - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle|^2 > C^2\varepsilon^2/4.$$

So $\|W_{\mathcal{P}_n}\|_2^2 > nC^2\varepsilon^2/4 \geq NC^2\varepsilon^2/4 > C^2$, which contradicts $\|W_{\mathcal{P}_n}\|_2 \leq \|W_{\mathcal{P}_n}\|_p \leq C$.

Case II: $1 < p < 2$. In this case, we no longer have an upper bound on $\|W_{\mathcal{P}_n}\|_2$ as before. We proceed by truncation: we stop the partition refinement process at step n , truncate the last step function, and then look back to calculate the energy

increment that would have come from doing the same partition refinement on the truncated graphon. Set

$$K := C(6/\varepsilon)^{1/(p-1)},$$

and define the truncation

$$U := W_{\mathcal{P}_n} 1_{|W_{\mathcal{P}_n}| \leq K}.$$

We claim that for $0 \leq i < n$,

$$(4.5) \quad \|U_{\mathcal{P}_{i+1}}\|_2^2 > \|U_{\mathcal{P}_i}\|_2^2 + (C\varepsilon/6)^2.$$

Then one has $\|U_{\mathcal{P}_n}\|_2^2 > n(C\varepsilon/6)^2 \geq N(C\varepsilon/6)^2 = C^2(6/\varepsilon)^{(2-p)/(p-1)}$, which contradicts

$$\begin{aligned} \|U_{\mathcal{P}_n}\|_2^2 &= \|W_{\mathcal{P}_n} 1_{|W_{\mathcal{P}_n}| \leq K}\|_2^2 \leq \left\| W_{\mathcal{P}_n} (K/|W_{\mathcal{P}_n}|)^{1-p/2} \right\|_2^2 \\ &= \|W_{\mathcal{P}_n}\|_p^p K^{2-p} \leq C^p K^{2-p} = C^2(6/\varepsilon)^{(2-p)/(p-1)}. \end{aligned}$$

It remains to prove (4.5). We have

$$\begin{aligned} \|W_{\mathcal{P}_n} - U\|_1 &= \|W_{\mathcal{P}_n} 1_{|W_{\mathcal{P}_n}| > K}\|_1 \\ &\leq \|W_{\mathcal{P}_n} (|W_{\mathcal{P}_n}|/K)^{p-1}\|_1 \\ &= \| |W_{\mathcal{P}_n}|^p \|_1 / K^{p-1} = \|W_{\mathcal{P}_n}\|_p^p / K^{p-1} \\ &\leq C^p / K^{p-1} = C\varepsilon/6. \end{aligned}$$

Since \mathcal{P}_n is a refinement of \mathcal{P}_i , we have $(W_{\mathcal{P}_n})_{\mathcal{P}_i} = W_{\mathcal{P}_i}$. So

$$(4.6) \quad \|W_{\mathcal{P}_i} - U_{\mathcal{P}_i}\|_1 = \|(W_{\mathcal{P}_n} - U)_{\mathcal{P}_i}\|_1 \leq \|W_{\mathcal{P}_n} - U\|_1 \leq C\varepsilon/6.$$

Similarly, $\|W_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_{i+1}}\|_1 \leq C\varepsilon/6$. Using the triangle inequality, (4.4), and (4.6), we find that

$$\begin{aligned} |\langle U_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| &\geq |\langle W_{\mathcal{P}_{i+1}} - W_{\mathcal{P}_i}, 1_{S' \times T'} \rangle| \\ &\quad - \|W_{\mathcal{P}_i} - U_{\mathcal{P}_i}\|_1 - \|W_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_{i+1}}\|_1 \\ &> C(\varepsilon/2 - \varepsilon/6 - \varepsilon/6) = C\varepsilon/6. \end{aligned}$$

Since \mathcal{P}_{i+1} is a refinement of \mathcal{P}_i , we have $\langle U_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_i}, U_{\mathcal{P}_i} \rangle = 0$. So by the Pythagorean theorem, followed by the Cauchy-Schwarz inequality, we have

$$\|U_{\mathcal{P}_{i+1}}\|_2^2 - \|U_{\mathcal{P}_i}\|_2^2 = \|U_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_i}\|_2^2 \geq |\langle U_{\mathcal{P}_{i+1}} - U_{\mathcal{P}_i}, 1_{S' \times T'} \rangle|^2 > (C\varepsilon/6)^2,$$

which proves (4.5), as desired. \square

This completes the proof of the weak regularity lemma for L^p upper regular graphons.

Remark 4.4. At the cost of slightly worse constants, the statement of Theorem 4.3 can be strengthened to provide an equipartition. To this end, we first apply the theorem to W , obtaining a partition \mathcal{P}_0 into at most 4^N parts such that each part has size at least η and $\|W - W_{\mathcal{P}_0}\|_{\square} \leq C\varepsilon$. Since W is assumed to be L^p upper regular, we obtain a graphon $U = W_{\mathcal{P}_0}$ such that $\|U\|_p \leq C$. Depending on whether $p \geq 2$ or $p \in (1, 2)$, we then apply Lemma 3.3 or Lemma 3.4 to U and the trivial partition of $[0, 1]$ consisting of the single class $[0, 1]$. As a consequence, for $k \geq 4^{\max\{10/\varepsilon^2, 10(3/\varepsilon)^{p/(p-1)}\}}$ we can find an equipartition \mathcal{P} of $[0, 1]$ into k parts such

that $\|W_{\mathcal{P}_0} - U_{\mathcal{P}}\|_{\square} = \|U - U_{\mathcal{P}}\|_{\square} \leq C\varepsilon$. With the help of the triangle inequality, this implies

$$\|W - U_{\mathcal{P}}\|_{\square} \leq 2C\varepsilon.$$

But $U_{\mathcal{P}}$ is a step functions with steps in \mathcal{P} , and it should approximate W at most as well as $W_{\mathcal{P}}$. While this is not quite true, it is true at the cost of another factor of two. To see this, we use the triangle inequality, $U_{\mathcal{P}} = (U_{\mathcal{P}})_{\mathcal{P}}$, and the fact that the stepping operator is a contraction with respect to the cut norm to bound

$$\begin{aligned} \|W - W_{\mathcal{P}}\|_{\square} &\leq \|W - U_{\mathcal{P}}\|_{\square} + \|W_{\mathcal{P}} - U_{\mathcal{P}}\|_{\square} \\ &= \|W - U_{\mathcal{P}}\|_{\square} + \|(W - U_{\mathcal{P}})_{\mathcal{P}}\|_{\square} \\ &\leq \|W - U_{\mathcal{P}}\|_{\square} + \|W - U_{\mathcal{P}}\|_{\square} \\ &= 2\|W - U_{\mathcal{P}}\|_{\square}. \end{aligned}$$

Putting everything together, we see that for any $k \geq 4^{\max\{10/\varepsilon^2, 10(3/\varepsilon)^{p/(p-1)}\}}$ we can find an equipartition \mathcal{P} of $[0, 1]$ into exactly k parts such that

$$\|W - W_{\mathcal{P}}\|_{\square} \leq 4C\varepsilon,$$

provided W is (C, η) -upper L^p regular with $\eta = 4^{-N-1}(\varepsilon/160)^{p/(p-1)}$, where $N = (6/\varepsilon)^{\max\{2, p/(p-1)\}}$.

Next we state the analogue of Theorem 4.3 for weighted graphs and explain how to modify the above proof to work for weighted graphs.

If G is a weighted graph, and $\mathcal{P} = \{V_1, \dots, V_m\}$ is a partition of $V(G)$, then we denote by $G_{\mathcal{P}}$ the weighted graph on $V(G)$ (with the same vertex weights as G) and edge weights as follows. For $s \in V_i, t \in V_j$ the edge between s and t is given weight

$$\beta_{st}(G_{\mathcal{P}}) = \sum_{x \in V_i, y \in V_j} \frac{\alpha_x \alpha_y}{\alpha_{V_i} \alpha_{V_j}} \beta_{xy}(G)$$

(note that we allow $x = y$). In other words, $G_{\mathcal{P}}$ is obtained from G by averaging the edge weights inside each $V_i \times V_j$. In terms of graphons, we have $W^{G_{\mathcal{P}}} = (W^G)_{\mathcal{P}}$, where we abuse notation by letting \mathcal{P} also denote the partition of $[0, 1]$ corresponding to the vertex partition.

Theorem 4.5 (Weak regularity lemma for L^p upper regular graphs). *Let $C > 0$, $p > 1$, and $0 < \varepsilon < 1$. Set $N = (6/\varepsilon)^{\max\{2, p/(p-1)\}}$ and $\eta = 4^{-N-1}(\varepsilon/320)^{p/(p-1)}$. Let $G = (V, E)$ be a (C, η) -upper L^p regular weighted graph. Then there exists a partition \mathcal{P} of V into at most 4^N parts, each having weight at least $\eta\alpha_G$, so that*

$$d_{\square} \left(\frac{G}{\|G\|_1}, \frac{G_{\mathcal{P}}}{\|G_{\mathcal{P}}\|_1} \right) \leq C\varepsilon.$$

Let us explain how one can modify the proofs in this section to prove Theorem 4.5. The only difference is that in the proceeding proofs, instead of taking arbitrary measurable sets, we are only allowed to take subsets of $[0, 1]$ corresponding to subsets of vertices. Another way to view this is that we are working with a different σ -algebra on $[0, 1]$, where the new σ -algebra comes from a partition of $[0, 1]$ into parts with measure equal to the vertex weights (as a fraction of the total vertex weights) of G . So previously in certain steps of the argument in Lemma 4.1 where we took an arbitrary subset S_1 a certain specified measure (say $\lambda(S_1) = \eta$), we have to be content with just having $\lambda(S_1) \in [\eta, 2\eta)$. This can be done since the

(C, η) -upper L^p regularity assumption implies no vertex occupies measure greater than η times the total vertex weight.

With this modification in place, Lemma 4.1 then becomes the following.

Lemma 4.6. *Assume $\eta < 1/13$. Let G be a (C, η) -upper L^p regular weighted graph with vertex weights α_i and edge weights β_{ij} . Let $S, T \subseteq V(G)$. If $\alpha_S \leq \delta \alpha_G$ for some $\delta \geq \eta$, then*

$$\left| \sum_{s \in S, t \in T} \beta_{st} \right| \leq 20\delta^{1-1/p} \sum_{i, j \in V(G)} |\beta_{ij}|.$$

The conclusion of Lemma 4.2 must be changed similarly, with the bound increased by a factor of 2. To prove Theorem 4.5 we can modify the proof of Theorem 4.3 to allow only subsets of vertices instead of arbitrary measurable sets.

Remark 4.7. As in Remark 4.4, we can achieve an equipartition in Theorem 4.5 at the cost of worse constants. Of course the indivisibility of vertices means we cannot always achieve an exact equipartition. Instead, by an *equipartition* of a graph G we mean a partition of $V(G)$ into k parts P_1, \dots, P_k such that for each i ,

$$\left| \alpha_{P_i} - \frac{\alpha_G}{k} \right| < \max_{j \in V(G)} \alpha_j.$$

The argument is the same as in Remark 4.4, except that we must use an equitable weak L^p regularity lemma for graphs, while Lemmas 3.3 and 3.4 were stated for graphons. For $p \geq 2$, Corollary 3.4(ii) in [8] supplies what we need, and exactly the same truncation argument used to derive Lemma 3.4 from Lemma 3.3 extends this argument to $p < 2$. The only difference is that the bound on η is now inherited from Theorem 4.5 instead of Theorem 4.3. We conclude that for $k \geq 4^{\max\{10/\varepsilon^2, 10(3/\varepsilon)^{p/(p-1)}\}}$, we can find an equipartition \mathcal{P} of $V(G)$ into exactly k parts such that

$$d_{\square} \left(\frac{G}{\|G\|_1}, \frac{G_{\mathcal{P}}}{\|G_{\mathcal{P}}\|_1} \right) \leq 4C\varepsilon,$$

provided G is (C, η) -upper L^p regular with $\eta = 4^{-N-1}(\varepsilon/320)^{p/(p-1)}$, where $N = (6/\varepsilon)^{\max\{2, p/(p-1)\}}$.

5. LIMIT OF AN L^p UPPER REGULAR SEQUENCE

Putting together the results in the last two sections, we obtain the limit for an L^p upper regular sequence, thereby completing the $(L^p$ upper regular sequence) \rightarrow $(L^p$ graphon limit) arrow in Figure 2.1.

Proof of Theorems 2.8 and 2.9. We give the proof of Theorem 2.9 (for graphons). The proof of Theorem 2.8 (for weighted graphs) is nearly identical (using Theorem 4.5 instead of Theorem 4.3).

Let W_n be an upper L^p regular sequence of graphons. In other words, there exists a sequence $\eta_n \rightarrow 0$ so that W_n is $(C + \eta_n, \eta_n)$ -upper L^p regular. Applying Theorem 4.3, we can find a sequence $\varepsilon_n \rightarrow 0$ so that for each n , there exists a partition \mathcal{P}_n of $[0, 1]$ for which each part has measure at least η_n and $\|W_n - (W_n)_{\mathcal{P}_n}\|_{\square} \leq \varepsilon_n$. We have $\|(W_n)_{\mathcal{P}_n}\|_p \leq C + \eta_n$ due to L^p upper regularity. By Theorem 2.13, there exists an L^p graphon W so that $\|W\|_p \leq C$ and $\delta_{\square}((W_n)_{\mathcal{P}_n}, W) \rightarrow 0$ along some subsequence. Since $\varepsilon_n \rightarrow 0$, $\delta_{\square}(W_n, W) \rightarrow 0$ along this subsequence. \square

The converse, Proposition 2.10, follows as an corollary of the following lemma. (Note that an L^p graphon W is automatically $(\|W\|_p, \eta)$ -upper L^p regular for every $\eta \geq 0$.)

Lemma 5.1. *Let $C > 0$, $\eta > 0$, and $1 \leq p \leq \infty$, and let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be a (C, η) -upper L^p regular graphon. Let $U: [0, 1]^2 \rightarrow \mathbb{R}$ be another graphon. If $\|W - U\|_{\square} \leq \eta^3$, then U is $(C + \eta, \eta)$ -upper L^p regular.*

Proof. For any subsets $S, T \subseteq [0, 1]$, we have $|\langle W - U, 1_{S \times T} \rangle| \leq \|W - U\|_{\square} \leq \eta^3$. It follows that

$$\left| \frac{1}{\lambda(S)\lambda(T)} \left(\int_{S \times T} W d\lambda - \int_{S \times T} U d\lambda \right) \right| \leq \frac{\eta^3}{\lambda(S)\lambda(T)} \leq \eta,$$

provided $\lambda(S), \lambda(T) \geq \eta$. So for any partition \mathcal{P} of $[0, 1]$ into sets each having measure at least η we have $|U_{\mathcal{P}} - W_{\mathcal{P}}| \leq \eta$ pointwise. Therefore,

$$\|U_{\mathcal{P}}\|_p \leq \| |W_{\mathcal{P}}| + \eta \|_p \leq \|W_{\mathcal{P}}\|_p + \|\eta\|_p \leq C + \eta.$$

It follows that U is $(C + \eta, \eta)$ -upper L^p regular. \square

Next we prove Proposition 2.12, which shows that without the L^p upper regularity assumption, a sequence of a graphs might not have a Cauchy subsequence (with respect to δ_{\square}). Furthermore, even a Cauchy sequence might not have a limit in the form of a graphon.

Proof of Proposition 2.12. (a) For each $n \geq 2$, let G_n be a graph on $n2^n$ vertices consisting of a single clique on n vertices. Then $\|G_n\|_1 = 2^{-2n}(n-1)/n$. Let $W_n = W^{G_n} / \|G_n\|_1$, where the support of W_n is contained in $[0, 2^{-n}]^2$. We claim that $\delta_{\square}(W_m, W_n) \geq 1/2$ for any $m \neq n$. Indeed, for any measure-preserving bijection $\sigma: [0, 1] \rightarrow [0, 1]$,

$$\begin{aligned} \|W_m - W_n^{\sigma}\|_{\square} &\geq \langle W_m - W_n^{\sigma}, 1_{[0, 2^{-m}]^2} \rangle \\ &\geq 1 - 2^{-2m} \|W_n\|_{\infty} \\ &= 1 - 2^{-2(m-n)} n / (n-1) \geq 1/2 \end{aligned}$$

for $m > n$.

(b) Our proof is inspired by a classic example of an L^1 martingale that converges almost surely but not in L^1 : a martingale that starts at 1 and then at each step either doubles or becomes zero. The analogue of this classic example will be a Cauchy sequence of graphs G_n whose normalized graphons converge to zero pointwise almost everywhere but not in cut distance. We will build this sequence inductively so that G_{n+1} is formed from G_n by replacing every edge of G_n with a quasi-random bipartite graph.

More precisely, for every n , let $\varepsilon_n = 4^{-n}$, and fix a simple graph H_n with $\delta_{\square}(H_n, 1_{[0, 1]^2}/2) \leq \varepsilon_n$. Let G_1 be the graph with one edge on two vertices. Set $G_{n+1} := G_n \times H_n$. In other words, to obtain G_{n+1} from G_n , replace every vertex v of G_n by $k = |V(H_n)|$ copies v_1, \dots, v_k . The edges of G_{n+1} consists of $u_i v_j$ where uv is an edge of G_n and ij is an edge of H_n .

Now we show that $(G_n)_{n \geq 0}$ is a Cauchy sequence with respect to the normalized cut metric. First, using the natural overlay between W^{G_n} and $W^{G_{n+1}}$ (the intervals

$I_1, \dots, I_{|V(G_n)|}$ corresponding to the vertices of G_n are each partitioned into $|H_n|$ parts corresponding to the vertices of G_{n+1} , we see that

$$\delta_{\square} \left(G_{n+1}, \frac{1}{2} G_n \right) \leq \left\| W^{G_{n+1}} - \frac{1}{2} W^{G_n} \right\|_{\square} \leq \left\| W^{H_n} - \frac{1}{2} 1_{[0,1]^2} \right\|_{\square} \leq \varepsilon_n,$$

since any $\langle W^{G_{n+1}} - W^{G_n}/2, 1_{A \times B} \rangle$ is equal to the sum of the contributions from each of the $|V(G_n)|^2$ cells $I_i \times I_j$, and the contribution from each cell is bounded by $\|W^{H_n} - 1_{[0,1]^2}/2\|_{\square} / |V(G_n)|^2$. Note that $\|G_{n+1}\|_1 / \|G_n\|_1 \in [1/2 - \varepsilon_n, 1/2 + \varepsilon_n]$. It follows that

$$\begin{aligned} \delta_{\square} \left(\frac{G_{n+1}}{\|G_{n+1}\|_1}, \frac{G_n}{\|G_n\|_1} \right) &= \frac{1}{\|G_{n+1}\|_1} \delta_{\square} \left(G_{n+1}, \frac{\|G_{n+1}\|_1}{\|G_n\|_1} G_n \right) \\ &\leq 3^{n+1} \left(\delta_{\square} \left(G_{n+1}, \frac{1}{2} G_n \right) + \varepsilon_n \right) \\ &\leq 3^{n+1} \cdot 2\varepsilon_n = 6 \cdot (3/4)^n. \end{aligned}$$

Thus the graphs $G_n / \|G_n\|_1$ form a Cauchy sequence with respect to δ_{\square} .

Next we show that $G_n / \|G_n\|_1$ does not converge to any graphon with respect to δ_{\square} . Let $W_n = W^{G_n} / \|G_n\|_1$ (properly aligned, so that the support of W_{n+1} is contained in the support of W_n). Then W_n converges to zero pointwise almost everywhere, but zero cannot be the δ_{\square} -limit of the sequence since $\mathbb{E}W_n = 1$ for all n . Indeed, as we will see shortly, there can be no U such that $\delta_{\square}(W_n, U) \rightarrow 0$. Assume by contradiction that there is such a graphon. Since W_n is non-negative, $\langle U, 1_{A \times B} \rangle \geq 0$ for every $A, B \subseteq [0, 1]$, implying that U is nonnegative as well. Furthermore $\mathbb{E}U = 1$, since $\mathbb{E}W_n = 1$ and $|\mathbb{E}W_n - \mathbb{E}U| \leq \delta_{\square}(U, W_n)$ (note that $\mathbb{E}U = \mathbb{E}U^{\sigma}$ for every measure-preserving bijection σ). We will show that U has the following property: for every $\varepsilon > 0$, there exists a subset $S \subseteq [0, 1]^2$ with $\lambda(S) \geq 1 - \varepsilon$ and $\langle U, 1_S \rangle \leq \varepsilon$. It would then follow that $U \equiv 0$, which is a contradiction.

Now it remains to verify the claim. There exists a sequence of measure-preserving bijections $\sigma_n: [0, 1] \rightarrow [0, 1]$ such that $\|W_n - U^{\sigma_n}\|_{\square} \rightarrow 0$. Fix an m with $\|G_m\|_1 \leq \varepsilon$, and let S be the complement of the support of W_m . So S is the disjoint union of at most $|V(G_m)|^2$ rectangles and $\lambda(S) \geq 1 - \varepsilon$. Choose an $n > m$ so that $\delta_{\square}(W_n, U) < |V(G_m)|^{-2} \varepsilon$. Since W_n is also zero on S , we have $\langle U^{\sigma_n}, 1_{A \times B} \rangle \leq \delta_{\square}(W_n, U) < |V(G_m)|^{-2} \varepsilon$ for every rectangle $A \times B$ contained in S . Summing over the at most $|V(G_m)|^2$ such rectangles whose disjoint union is S , we find that $\langle U^{\sigma_n}, 1_S \rangle \leq \varepsilon$. The claim then follows. \square

The following proposition shows that when dealing with graphs, we can replace the measure-preserving bijection implicit in δ_{\square} with a permutation of the vertices.

Proposition 5.2. *Let $C > 0$ and $p > 1$, and let $(G_n)_{n \geq 0}$ be a C -upper L^p regular sequence of weighted graphs such that $\delta_{\square}(G_n / \|G_n\|_1, U) \rightarrow 0$ for some L^p graphon U . Then the vertices of the graphs G_n may be ordered in such a way that $\|W^{G_n} / \|G_n\|_1 - U\|_{\square} \rightarrow 0$.*

We recall the following lemma⁵ from [25, Theorem 9.29], where it is attributed to Alon. Here $\hat{\delta}_{\square}(G_1, G_2)$ denotes the cut distance with respect to the optimal *integral*

⁵In [25], Theorem 9.29 is stated for weighted graphs whose edge weights lie in $[0, 1]$, but it immediately implies the version stated here.

overlay, i.e., $\hat{\delta}_\square(G_1, G_2) := \min_{G'_1} d_\square(G'_1, G_2)$, where G'_1 is G_1 with any reordering of its vertices (assuming $|V(G_1)| = |V(G_2)|$).

Lemma 5.3. *For any two weighted graphs G_1 and G_2 with the same number v of vertices, unit node weights, and edge weights in $[-1, 1]$,*

$$\hat{\delta}_\square(G_1, G_2) \leq \delta_\square(G_1, G_2) + \frac{34}{\sqrt{\log v}}.$$

As an immediate corollary, if the graphs in the lemma have edge weights in $[-K, K]$ instead for some $K > 0$, then the same inequality holds with the final term replaced by $34K/\sqrt{\log v}$.

Note that it was proved in [8, Theorem 2.3] that

$$\delta_\square(G_1, G_2) \leq \hat{\delta}_\square(G_1, G_2) \leq 32\delta_\square(G_1, G_2)^{1/67}$$

under the hypotheses of Lemma 5.3. It remains open whether $\hat{\delta}_\square(G_1, G_2) = O(\delta_\square(G_1, G_2))$, which would slightly simplify the proof of Proposition 5.2 if true.

Proof of Proposition 5.2. Let $W_n = W^{G_n} / \|G_n\|_1$, which depends on the ordering of the vertices of G_n . We need to show that some such ordering of vertices yields $d_\square(W_n, U) \rightarrow 0$, given that $\delta_\square(W_n, U) \rightarrow 0$.

First we prove the lemma by a truncation argument under the additional hypotheses that the graphs G_n all have unit node weights and $\|W_n\|_p \leq C$. We begin by choosing a sequence of truncations K_n so that $K_n \rightarrow \infty$ and $K_n/\sqrt{\log |V(G_n)|} \rightarrow 0$. (Note that $|V(G_n)| \rightarrow \infty$ because $(G_n)_{n \geq 0}$ is a C -upper L^p regular sequence.)

Let U_n denote the step function $U_{\mathcal{P}_n}$, where \mathcal{P}_n is the partition of $[0, 1]$ into $|V(G_n)|$ equal length intervals. By Lemma 5.3, we can reorder the vertices of G_n so that the corresponding graphon W_n satisfies

$$\begin{aligned} d_\square(W_n 1_{|W_n| \leq K_n}, U_n 1_{|U_n| \leq K_n}) &\leq \delta_\square(W_n 1_{|W_n| \leq K_n}, U_n 1_{|U_n| \leq K_n}) + \frac{34K_n}{\sqrt{\log |V(G_n)|}} \\ &\leq \delta_\square(W_n, W_n 1_{|W_n| \leq K_n}) + \delta_\square(U_n 1_{|U_n| \leq K_n}, U) \\ &\quad + \delta_\square(W_n, U) + \frac{34K_n}{\sqrt{\log |V(G_n)|}}. \end{aligned}$$

Using this inequality to bound the right side of

$$\begin{aligned} d_\square(W_n, U) &\leq d_\square(W_n, W_n 1_{|W_n| \leq K_n}) + d_\square(U_n 1_{|U_n| \leq K_n}, U) \\ &\quad + d_\square(W_n 1_{|W_n| \leq K_n}, U_n 1_{|U_n| \leq K_n}) \end{aligned}$$

and bounding δ_\square by d_\square yields

$$\begin{aligned} d_\square(W_n, U) &\leq \delta_\square(W_n, U) + \frac{34K_n}{\sqrt{\log |V(G_n)|}} \\ &\quad + 2d_\square(W_n, W_n 1_{|W_n| \leq K_n}) + 2d_\square(U_n 1_{|U_n| \leq K_n}, U). \end{aligned}$$

To estimate $2d_\square(W_n, W_n 1_{|W_n| \leq K_n}) + 2d_\square(U_n 1_{|U_n| \leq K_n}, U)$, we will bound d_\square by d_1 . We have

$$\begin{aligned} d_1(W_n, W_n 1_{|W_n| \leq K_n}) &\leq \|W_n 1_{|W_n| > K_n}\|_1 \\ &\leq \|W_n (|W_n|/K_n)^{p-1}\|_1 = \|W_n\|_p^p / K_n^{p-1} \leq C^p / K_n^{p-1}. \end{aligned}$$

Similarly,

$$d_1(U_n, U_n 1_{|U_n| \leq K_n}) \leq \|U_n\|_p^p / K_n^{p-1} \leq \|U\|_p^p / K_n^{p-1} \leq C^p / K_n^{p-1}$$

(note that $\|U\|_p \leq C$ by Theorem 2.8). It follows that

$$d_{\square}(W_n, U) \leq \delta_{\square}(W_n, U) + \frac{34K_n}{\sqrt{\log |V(G_n)|}} + \frac{2C^p}{K_n^{p-1}} + 2d_1(U, U_n).$$

We have $d_1(U, U_n) \rightarrow 0$ by the Lebesgue differentiation theorem, and all the other terms tend to zero by assumption. Thus $d_{\square}(W_n, U) \rightarrow 0$.

Next we relax the assumption of unit node weights, and instead assume that every vertex in G_n has weight $1+o(|V(G_n)|^{-1})$ (i.e., nearly equal node weights). Let \widetilde{W}_n be a step function with the same values as W_n , but where the step widths have all been modified to be exactly $1/|V(G_n)|$. We will show that $\|\widetilde{W}_n - W_n\|_1 = o(1)$, which suffices to reduce this case to the previous one. Indeed, suppose the step widths of W_n are all in the interval $[1/|V(G_n)| - \alpha_n, 1/|V(G_n)| + \alpha_n]$, where $\alpha_n |V(G_n)|^2 \rightarrow 0$. Then \widetilde{W}_n and W_n differ on a set B_n of measure at most $2|V(G_n)|^2 \alpha_n = o(1)$, because each of the lines separating the steps is moved by less than $|V(G_n)| \alpha_n$ (typically much less). By Hölder's inequality,

$$\|\widetilde{W}_n - W_n\|_1 = \int_{B_n} |\widetilde{W}_n - W_n| d\lambda \leq \|\widetilde{W}_n - W_n\|_p \lambda(B_n)^{1-1/p}.$$

We know that $\|W_n\|_p \leq C$, and it is easy to check that $\|\widetilde{W}_n\|_p$ is bounded as well.

Since $\lambda(B_n) \rightarrow 0$, it follows that $\|\widetilde{W}_n - W_n\|_1 \rightarrow 0$. This reduces the case of nearly equal node weights to that of equal node weights.

Finally, we prove the result for a C -upper L^p regular sequence of weighted graphs. We may replace C by a larger value if necessary and assume that G_n is (C, η_n) -upper L^p regular with $\eta_n \rightarrow 0$. Let $\alpha_{\max}(G_n)$ denote the largest node weight in G_n , and recall that $\alpha_{\max}(G_n)/\alpha_{G_n} \leq \eta_n$ by Definition 2.1. By Remark 4.7, there is some equipartition \mathcal{P}_n of $V(G_n)$ into k_n parts, for some slowly growing k_n satisfying $k_n \rightarrow \infty$ and $k_n^2 \eta_n \rightarrow 0$, so that $W'_n := (W_n)_{\mathcal{P}_n}$ satisfies $d_{\square}(W'_n, W_n) \rightarrow 0$. Then $\delta_{\square}(W'_n, U) \rightarrow 0$. Furthermore, $\|W'_n\|_p \leq C$ since G_n is (C, η_n) -upper L^p regular. Note that W'_n is a step function with step widths $1/k_n + o(1/k_n^2)$, since \mathcal{P}_n is an equipartition into k_n parts and $\alpha_{\max}(G_n)/\alpha_G = o(1/k_n^2)$. Now we apply the case in the previous paragraph to W'_n to reorder the parts of \mathcal{P}_n so that $d_{\square}(W'_n, U) \rightarrow 0$. If we order the vertices of G_n according to this ordering of \mathcal{P}_n and arbitrarily order the vertices within each part, then $d_{\square}(W_n, U) \leq d_{\square}(W'_n, W_n) + d_{\square}(W'_n, U) \rightarrow 0$, as desired. \square

6. W -RANDOM WEIGHTED GRAPHS

In this section and the next, we prove Theorem 2.14 on W -random graphs, thereby traversing the outer arrows of Figure 2.1. First, in this section, we address the arrow (L^p graphon limit) \rightarrow (L^p graphon sequence) by proving Theorem 2.14(a), which says that $d_1(\mathbf{H}(W, n), W) \rightarrow 1$ almost surely (i.e., with probability 1) for any L^1 graphon W .

The following theorem of Hoeffding on U -statistics implies that $\|\mathbf{H}(W, n)\|_1 \rightarrow \|W\|_1$ almost surely.

Theorem 6.1 (Hoeffding [20]). *Let $W: [0, 1]^2 \rightarrow \mathbb{R}$ be a symmetric, integrable function, and let x_1, x_2, \dots be a sequence of i.i.d. random variables uniformly*

chosen from $[0, 1]$. Then with probability 1,

$$\lim_{n \rightarrow \infty} \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} W(x_i, x_j) \rightarrow \int_{[0,1]^2} W(x, y) dx dy.$$

Proof of Theorem 2.14(a). All weighted random graphs $\mathbf{H}(\cdot, n)$ in this proof come from the same random sequence x_1, x_2, \dots with terms drawn uniformly i.i.d. from $[0, 1]$.

Fix $\varepsilon > 0$. It suffices to show that $\limsup_{n \rightarrow \infty} d_1(\mathbf{H}(W, n), W) \leq \varepsilon$ holds with probability 1.

Let \mathcal{P} denote the partition of $[0, 1]$ into m equal intervals, where m is chosen to be sufficiently large that $\|W - W_{\mathcal{P}}\|_1 \leq \varepsilon/2$. Fix this m and \mathcal{P} . Since the sequence x_1, x_2, \dots is equidistributed among the m intervals of \mathcal{P} , with probability 1 we have $d_1(\mathbf{H}(W_{\mathcal{P}}, n), W_{\mathcal{P}}) \rightarrow 0$ as $n \rightarrow \infty$.

We have $d_1(\mathbf{H}(W, n), \mathbf{H}(W_{\mathcal{P}}, n)) = \|\mathbf{H}(W - W_{\mathcal{P}}, n)\|_1$, which by Theorem 6.1 converges almost surely to $\|W - W_{\mathcal{P}}\|_1$. It follows that, with probability 1, the limit superior (as $n \rightarrow \infty$) of

$$d_1(\mathbf{H}(W, n), W) \leq d_1(W, W_{\mathcal{P}}) + d_1(W_{\mathcal{P}}, \mathbf{H}(W_{\mathcal{P}}, n)) + d_1(\mathbf{H}(W_{\mathcal{P}}, n), \mathbf{H}(W, n))$$

is at most $2\|W - W_{\mathcal{P}}\|_1 \leq \varepsilon$, as claimed. \square

7. SPARSE RANDOM GRAPHS

In this section we prove Theorem 2.14(b); i.e., we prove that with probability 1, $d_{\square}(\rho_n^{-1}\mathbf{G}(n, W, \rho_n), W) \rightarrow 0$. From Theorem 2.14(a) we know that $\lim_{n \rightarrow \infty} d_1(\mathbf{H}(n, W), W) = 0$ with probability 1. So it remains to show that

$$(7.1) \quad d_{\square}(\rho_n^{-1}\mathbf{G}(n, W, \rho_n), \mathbf{H}(n, W)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Here $\mathbf{G}(n, W, \rho_n)$ and $\mathbf{H}(n, W)$ are both generated from a common i.i.d. random sequence $x_1, x_2, \dots \in [0, 1]$. We keep this assumption throughout the section.

We will need the following variant of the Chernoff bound. The proof (a modification of the usual proof) is included in Appendix B.

Lemma 7.1. *Let X_1, \dots, X_n be independent random variables, where for each i , X_i is distributed as either Bernoulli(p_i) or $-$ Bernoulli(p_i). Let $X = X_1 + \dots + X_n$ and $q = p_1 + \dots + p_n$. Then for every $\lambda > 0$,*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \lambda q) \leq \begin{cases} 2 \exp(-\frac{1}{3}\lambda^2 q) & \text{if } 0 < \lambda \leq 1, \\ 2 \exp(-\frac{1}{3}\lambda q) & \text{if } \lambda > 1. \end{cases}$$

For a weighted graph H with unit vertex weights and edge weights $\beta_{ij} \in [-1, 1]$, we use $\mathbf{G}(H)$ to denote the random graph with vertex set $V(H)$ and an edge between i and j with probability $|\beta_{ij}|$, and we assign the edge weight $+1$ if $\beta_{ij} > 0$ and -1 if $\beta_{ij} < 0$. (In other words, $\mathbf{G}(H) = \mathbf{G}(H, 1)$ in the notation of §2.7.)

The next two lemmas form the (L^p graphon sequence) \rightarrow (L^p upper regular sequence) arrow in Figure 2.1.

Lemma 7.2. *Let $\varepsilon > 0$. Let H be a weighted graph on n vertices with unit vertex weights, edge weights $\beta_{ij}(H) \in [-1, 1]$, and $\beta_{ii}(H) = 0$ for all $i, j \in V(H)$. Then*

$$\mathbb{P}(d_{\square}(\mathbf{G}(H), H) \leq \varepsilon \|H\|_1) \geq 1 - 2^{n+1} \exp\left(-\frac{1}{24} \min\{\varepsilon, \varepsilon^2\} \|H\|_1 n^2\right).$$

Proof. Let $V = V(G) = V(H) = [n]$. For any subset $U \subseteq V$, let

$$\beta_U(H) = \sum_{\substack{i < j \\ i, j \in U}} \beta_{ij}(H)$$

be the sum of the edge weights of H inside U . Similarly define $\beta_U(G)$, where $G = \mathbf{G}(H)$. We also define

$$|\beta|_U(H) = \sum_{\substack{i < j \\ i, j \in U}} |\beta_{ij}(H)|.$$

Set

$$\lambda = \frac{\varepsilon n^2 \|H\|_1}{4 |\beta|_U(H)} \geq \frac{\varepsilon n^2 \|H\|_1}{4 |\beta|_V(H)} = \frac{\varepsilon}{2}.$$

It follows from Lemma 7.1 that

$$\begin{aligned} \mathbb{P} \left(|\beta_U(G) - \beta_U(H)| \geq \frac{1}{4} \varepsilon n^2 \|H\|_1 \right) &= \mathbb{P} (|\beta_U(G) - \beta_U(H)| \geq \lambda |\beta|_U(H)) \\ &\leq 2 \exp \left(-\frac{1}{3} \min\{\lambda, 1\} \lambda |\beta|_U(H) \right) \\ &\leq 2 \exp \left(-\frac{1}{12} \min \left\{ \frac{\varepsilon}{2}, 1 \right\} \varepsilon n^2 \|H\|_1 \right) \\ &\leq 2 \exp \left(-\frac{1}{24} \min\{\varepsilon^2, \varepsilon\} n^2 \|H\|_1 \right). \end{aligned}$$

By the union bound, with probability at least $1 - 2^{n+1} \exp(-\frac{1}{24} \min\{\varepsilon^2, \varepsilon\} n^2 \|H\|_1)$,

$$(7.2) \quad |\beta_U(G) - \beta_U(H)| \leq \frac{1}{4} \varepsilon n^2 \|H\|_1 \quad \text{for all } U \subseteq [n].$$

For $S, T \subseteq V$, let

$$\beta_{S \times T} = \sum_{s \in S, t \in T} \beta_{st}.$$

We have

$$\beta_{S \times T} = \beta_{S \cup T} + \beta_{S \cap T} - \beta_{S \setminus T} - \beta_{T \setminus S}.$$

We deduce from (7.2) that

$$|\beta_{S \times T}(G) - \beta_{S \times T}(H)| \leq \varepsilon n^2 \|H\|_1 \quad \text{for all } S, T \subseteq [n],$$

which is equivalent to $d_{\square}(G, H) \leq \varepsilon \|H\|_1$. \square

The following lemma shows that $d_{\square}(\rho_n^{-1} \mathbf{G}(H_n, \rho_n), H_n) \rightarrow 0$ for any sequence of weighted graphs that satisfy certain mild conditions on the edge weights. Recall the definition of the random graph $\mathbf{G}(H_n, \rho_n)$ from §2.7.

Lemma 7.3. *Let $\rho_n > 0$ with $\rho_n \rightarrow 0$ and $n\rho_n \rightarrow \infty$. For each n let H_n be a weighted graph with n vertices all with unit vertex weights, and containing no loops. Suppose that $\|H_n\|_1$ is uniformly bounded and the edge weights $\beta_{ij}(H)$ satisfy*

$$(7.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \max\{|\beta_{ij}(H_n)| - \rho_n^{-1}, 0\} = 0.$$

Then

$$\lim_{n \rightarrow \infty} d_{\square}(\rho_n^{-1} \mathbf{G}(H_n, \rho_n), H_n) = 0$$

with probability 1.

Proof. Define the weighted graph H'_n with edge weights

$$\beta_{ij}(H'_n) = \text{sign}(\beta_{ij}(H_n)) \min\{\rho_n |\beta_{ij}(H_n)|, 1\}.$$

So $\mathbf{G}(H_n, \rho_n) = \mathbf{G}(H'_n)$. We have

$$\begin{aligned} (7.4) \quad d_1(\rho_n^{-1}H'_n, H_n) &= \frac{1}{n^2} \sum_{i,j=1}^n |\rho_n^{-1}\beta_{ij}(H'_n) - \beta_{ij}(H_n)| \\ &= \frac{1}{n^2} \sum_{i,j=1}^n \max\{|\beta_{ij}(H_n)| - \rho_n^{-1}, 0\}, \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$, by assumption (7.3). It follows that $\rho_n^{-1}\|H'_n\|_1 = \|H_n\|_1 + o(1) = O(1)$, as we assumed that $\|H_n\|_1$ is uniformly bounded. By Lemma 7.2 for every $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}(d_{\square}(\mathbf{G}(H'_n), H'_n) \leq \varepsilon \rho_n) &\geq 1 - 2^{n+1} \exp\left(-\frac{1}{24} \min\left\{\frac{\varepsilon \rho_n}{\|H'_n\|_1}, 1\right\} \varepsilon \rho_n n^2\right) \\ &\geq 1 - 2^{n+1} \exp\left(-\frac{1}{24} \min\{\Omega(\varepsilon), 1\} \varepsilon \rho_n n^2\right) \\ &\geq 1 - 2^{-\omega(n)} \end{aligned}$$

as $n \rightarrow \infty$, since $n\rho_n \rightarrow \infty$. So by the Borel-Cantelli lemma,

$$\lim_{n \rightarrow \infty} \rho_n^{-1} d_{\square}(\mathbf{G}(H'_n), H'_n) = 0$$

with probability 1. Combined with (7.4) we obtain the desired conclusion. \square

Finally we put everything together and complete Figure 2.1 with the arrow (L^p graphon limit) \rightarrow (L^p upper regular sequence).

Proof of Theorem 2.14(b). We need to show (7.1). We apply Lemma 7.3 with $H_n = \mathbf{H}(W, n)$. By Theorem 6.1, $\|H_n\|_1 \rightarrow \|W\|_1$ almost surely, so in particular $\|H_n\|_1$ is uniformly bounded. It remains to check (7.3). We have

$$\frac{1}{n^2} \sum_{1 \leq i < j \leq n} \max\{|\beta_{ij}(H_n)| - \rho_n^{-1}, 0\} = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \max\{|W(x_i, x_j)| - \rho_n^{-1}, 0\},$$

which converges to 0 as $n \rightarrow \infty$ with probability 1 by Theorem 6.1. Indeed, since $\rho_n \rightarrow 0$, for every $K > 0$ the limit superior of the above expression is bounded by $\frac{1}{2}\|\max\{|W| - K, 0\}\|_1$ by Theorem 6.1, and this can be made arbitrarily small by choosing K large. \square

Proof of Corollary 2.15. By Theorem 2.14(b), $\delta_{\square}(\rho_n^{-1}G_n, W) \rightarrow 0$ with probability 1 as $n \rightarrow \infty$, and applying the theorem to $|W|$ shows that $\rho_n^{-1}\|G_n\|_1 \rightarrow \|W\|_1$ with

probability 1. It follows that

$$\begin{aligned} \delta_{\square} \left(\frac{G_n}{\|G_n\|_1}, \frac{W}{\|W\|_1} \right) &= \frac{\rho_n}{\|G_n\|_1} \delta_{\square} \left(\rho_n^{-1} G_n, \frac{\|G_n\|_1}{\rho_n \|W\|_1} W \right) \\ &\leq \frac{\rho_n}{\|G_n\|_1} \left(\delta_{\square}(\rho_n^{-1} G_n, W) + \delta_{\square} \left(W, \frac{\|G_n\|_1}{\rho_n \|W\|_1} W \right) \right) \\ &\leq \frac{\rho_n}{\|G_n\|_1} \left(\delta_{\square}(\rho_n^{-1} G_n, W) + \|W\|_{\square} \left| 1 - \frac{\|G_n\|_1}{\rho_n \|W\|_1} \right| \right) \\ &\rightarrow 0, \end{aligned}$$

as desired. \square

Proof of Proposition 2.16. By Corollary 2.11, the sequence $(G_n)_{n \geq 0}$ must be $\|W\|_p$ -upper L^p regular. From $\delta_{\square}(G_n/\|G_n\|_1, W) \rightarrow 0$ we obtain $\|W\|_1 = 1$ (note that $W \geq 0$ because G_n is simple), and by Proposition A.1 we have $n \|G\|_1 \rightarrow \infty$. It then follows from Corollary 2.15 that $\delta_{\square}(G'_n/\|G'_n\|_1, W) \rightarrow 0$ with probability 1. By Proposition 5.2 we can order the vertices of G_n and G'_n so that $d_{\square}(G_n/\|G_n\|_1, W) \rightarrow 0$ and $d_{\square}(G'_n/\|G'_n\|_1, W) \rightarrow 0$, and thus

$$d_{\square} \left(\frac{G_n}{\|G_n\|_1}, \frac{G'_n}{\|G'_n\|_1} \right) \rightarrow 0,$$

as desired. \square

8. COUNTING LEMMA FOR L^p GRAPHONS

In this section we establish results relating to counting lemmas for L^p graphons, as stated in §2.9.

We use the following generalization of Hölder's inequality from [17] (also see [28, Theorem 3.1]). This inequality played a key role in recent work by the fourth author and Lubetzky [28] resolving a conjecture of Chatterjee and Varadhan [10] on large deviations in random graphs, which involves an application of graph limits.

Theorem 8.1 (Generalized Hölder's inequality). *Let μ_1, \dots, μ_n be probability measures on $\Omega_1, \dots, \Omega_n$, respectively, and let $\mu = \prod_{i=1}^n \mu_i$ be the product measure on $\Omega = \prod_{i=1}^n \Omega_i$. Let A_1, \dots, A_m be nonempty subsets of $[n] := \{1, \dots, n\}$ and write $\Omega_A = \prod_{\ell \in A} \Omega_{\ell}$ and $\mu_A = \prod_{\ell \in A} \mu_{\ell}$. Let $f_i \in L^{p_i}(\Omega_{A_i}, \mu_{A_i})$ with $p_i \geq 1$ for each $i \in [m]$ and suppose in addition that $\sum_{i: \ell \in A_i} (1/p_i) \leq 1$ for each $\ell \in [n]$. Then*

$$\int \prod_{i=1}^m |f_i| \, d\mu \leq \prod_{i=1}^m \left(\int |f_i|^{p_i} \, d\mu_{A_i} \right)^{1/p_i}.$$

Proof of Proposition 2.19. For the first assertion, we can give an example in the form of a separable graphon, i.e., one of the form $W(x, y) = w(x)w(y)$. Let $w: [0, 1] \rightarrow [0, \infty)$ be in $L^p([0, 1])$ for all $p < \Delta$ but not $p = \Delta$, e.g., $w(x) = x^{-1/\Delta}$ (and $w(0) = 0$). Then $\|W\|_p = \|w\|_p^2 < \infty$ for all $p < \Delta$, but $t(F, W) = \prod_{v \in V(G)} \|w\|_{\deg_F(v)}^{\deg_F(v)}$, which is infinite since $\|w\|_{\Delta} = \infty$.

For the second assertion, apply Theorem 8.1 with $n = |V(F)|$, $\Omega_i = [0, 1]$, μ_i equal to Lebesgue measure, A_1, \dots, A_m the edges of F (i.e., they are two-element subsets of $V(F)$), and $p_i = \Delta$ for all i . \square

Lemma 8.2. *Let F be a simple graph with maximum degree Δ . Let $\Delta < p < \infty$ and let $q = p/(p - \Delta + 1)$. For each edge $e \in E(F)$, let W_e be an L^p graphon. Fix an edge $e_1 \in E(F)$. Then*

$$\left| \int_{[0,1]^{|V(F)|}} \prod_{ij \in E(F)} W_{ij}(x_i, x_j) dx_1 \cdots dx_{|V(F)|} \right| \leq \|W_{e_1}\|_q \prod_{e \in E(F) \setminus \{e_1\}} \|W_e\|_p.$$

Proof. Apply Theorem 8.1 with $n = |V(F)|$, $\Omega_i = [0, 1]$, μ_i equal to Lebesgue measure, A_1, \dots, A_m the edges of F (with $A_1 = e_1$), $p_1 = q$, and $p_i = p$ for $i \geq 2$. The inequality $\sum_{i: \ell \in A_i} (1/p_i) \leq 1$ is satisfied for each ℓ because $q < p$ and $1/q + (\Delta - 1)/p = 1$ (at most one term $1/p_i$ with $\ell \in A_i$ can equal $1/q$, the others equal $1/p$, and there are at most Δ terms). \square

Proof of Theorem 2.20. Let $V(F) = \{1, 2, \dots, n\}$ and $E(F) = \{e_1, \dots, e_m\}$. Let i_t, j_t be the endpoints of e_t , for $1 \leq t \leq m$. We may assume that $\|U - W\|_{\square} \leq \varepsilon$. We have

$$\begin{aligned} t(F, U) - t(F, W) &= \int_{[0,1]^n} \left(\prod_{t=1}^m U(x_{i_t}, x_{j_t}) - \prod_{t=1}^m W(x_{i_t}, x_{j_t}) \right) dx_1 \cdots dx_n. \\ &= \sum_{t=1}^m \int_{[0,1]^n} \left(\prod_{s < t} U(x_{i_s}, x_{j_s}) \right) (U(x_{i_t}, x_{j_t}) - W(x_{i_t}, x_{j_t})) \cdot \\ &\quad \left(\prod_{s > t} W(x_{i_s}, x_{j_s}) \right) dx_1 \cdots dx_n. \end{aligned}$$

It suffices to show that for each $t = 1, \dots, m$,

$$(8.1) \quad \left| \int_{[0,1]^n} \left(\prod_{s < t} U(x_{i_s}, x_{j_s}) \right) (U(x_{i_t}, x_{j_t}) - W(x_{i_t}, x_{j_t})) \left(\prod_{s > t} W(x_{i_s}, x_{j_s}) \right) dx_1 \cdots dx_n \right| \leq 2(m-1+p-\Delta) \left(\frac{2\varepsilon}{p-\Delta} \right)^{\frac{p-\Delta}{p-\Delta+m-1}}.$$

Let $K > 0$, which we will choose later. Let $U = U_{\leq K} + U_{> K}$, where $U_{\leq K} := U1_{|U| \leq K}$ and $U_{> K} := U1_{|U| > K}$. Similarly, let $W_{\leq K} := W1_{|W| \leq K}$ and $W_{> K} := W1_{|W| > K}$. We claim that

$$(8.2) \quad \left| \int_{[0,1]^n} \left(\prod_{s < t} U_{\leq K}(x_{i_s}, x_{j_s}) \right) (U(x_{i_t}, x_{j_t}) - W(x_{i_t}, x_{j_t})) \cdot \left(\prod_{s > t} W_{\leq K}(x_{i_s}, x_{j_s}) \right) dx_1 \cdots dx_n \right| \leq 4K^{m-1}\varepsilon.$$

Indeed, if we fix the value of x_i for all $i \in [n] \setminus \{i_t, j_t\}$, then the integral in (8.2) has the form

$$(8.3) \quad K^{m-1} \int_{[0,1]^2} (U(x_{i_t}, x_{j_t}) - W(x_{i_t}, x_{j_t})) a(x_{i_t}) b(x_{j_t}) dx_{i_t} dx_{j_t}$$

for some functions $a(\cdot)$ and $b(\cdot)$ with $\|a\|_{\infty}, \|b\|_{\infty} \leq 1$, where $a(\cdot)$ and $b(\cdot)$ depend on the values of x_i for $i \in [n] \setminus \{i_t, j_t\}$ that we fixed. Thus (8.3) is bounded in

absolute value by $K^{m-1} \|U - W\|_{\infty \rightarrow 1} \leq 4K^{m-1}\varepsilon$, using (2.3). The inequality (8.2) then follows.

Next we claim that the difference between the integral in (8.1) and the integral in (8.2) is bounded in absolute value by $2(m-1)/K^{p-\Delta}$. Indeed, writing this difference as a telescoping sum in a similar fashion to what we did at the beginning of this proof, it suffices to show that each expression of the following form is bounded in absolute value by $2/K^{p-\Delta}$:

$$(8.4) \quad \int_{[0,1]^n} \left(\prod_{s < t} U_*(x_{i_s}, x_{j_s}) \right) (U(x_{i_t}, x_{j_t}) - W(x_{i_t}, x_{j_t})) \left(\prod_{s > t} W_*(x_{i_s}, x_{j_s}) \right) dx_1 \cdots dx_n,$$

where we replace exactly one of the $m-1$ subscript $*$'s by ' $> K$ ', replace some of the other $*$'s by ' $\leq K$ ', and then erase the remaining $*$'s. Now we apply Lemma 8.2 with the special edge e_0 corresponding to the factor whose subscript is replaced by ' $> K$ '. We use $\|U_{\leq K}\|_p \leq \|U\|_p \leq 1$ and $\|W_{\leq K}\|_p \leq \|W\|_p \leq 1$. Using the triangle inequality we have $\|U - W\|_p \leq 2$. Also,

$$\|U_{> K}\|_q \leq \left\| U(|U|/K)^{p/q-1} \right\|_q = \|U\|_p^{p/q} / K^{p/q-1} \leq 1/K^{p-\Delta}.$$

It then follows from Lemma 8.2 that an integral of the form (8.4) is at most $2/K^{p-\Delta}$ in absolute value.

Combining the bounds on (8.2) and (8.4), we see that the integral in (8.1) is bounded in absolute value by

$$4K^{m-1}\varepsilon + 2(m-1)/K^{p-\Delta}.$$

We optimize this bound by choosing $K = ((p-\Delta)/(2\varepsilon))^{1/(m-1+p-\Delta)}$, which gives the bound in (8.1) that we claimed. \square

Next we give an example showing that no counting lemma can hold when $p \leq \Delta$.

Proof of Proposition 2.22. By nesting of norms, we only need to consider the case $p = \Delta$. For each $n \geq 1$, consider the separable graphon W_n defined by

$$W_n(x, y) := w_n(x)w_n(y),$$

where $w_n(x) := 1 + u_n(x)$ with $u_n(x) := (x \ln n)^{-1/\Delta} \mathbf{1}_{[1/n, 1]}(x)$. We chose u_n so that it satisfies $\|u_n\|_\Delta = 1$ and $\lim_{n \rightarrow \infty} \|u_n\|_p = 0$ for $1 \leq p < \Delta$.

We have

$$\|W_n\|_\Delta = \|w_n\|_\Delta^2 \leq (1 + \|u_n\|_\Delta)^2 = 4.$$

Also, since $W_n(x, y) - 1 = u_n(x) + u_n(y) + u_n(x)u_n(y)$,

$$\|W_n - 1\|_1 \leq 2\|u_n\|_1 + \|u_n\|_1^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It remains to verify that $\liminf_{n \rightarrow \infty} t(F, W_n) > 1$. Since W_n is separable,

$$t(F, W_n) = \prod_{v \in V(F)} \|w_n\|_{\deg_F(v)}^{\deg_F(v)}.$$

For any integer k ,

$$\|w_n\|_k^k = \mathbb{E}[(1 + u_n)^k] = \sum_{i=0}^k \binom{k}{i} \|u_n\|_i^i.$$

Since $\|u_n\|_\Delta = 1$ and $\lim_{n \rightarrow \infty} \|u_n\|_p = 0$ for any $1 \leq p < \Delta$, we find that $\lim_{n \rightarrow \infty} \|w_n\|_k^k = 1$ when $1 \leq k < \Delta$, and $\lim_{n \rightarrow \infty} \|w_n\|_\Delta^\Delta = 2$. Therefore,

$$\lim_{n \rightarrow \infty} t(F, W_n) = 2^{|\{v \in V(G) : \deg_F(v) = \Delta\}|} > 1,$$

as desired. \square

There has been some recent work by the fourth author along with Conlon and Fox [14, 15] developing counting lemmas for sparse graphs assuming additional hypotheses. Namely one assumes that the sparse graph G is a relatively dense subgraph of another sparse graph Γ that has certain pseudorandomness properties. For example, to obtain a counting lemma for K_3 in G , one assumes that $t(H, \Gamma / \|\Gamma\|_1) = 1 + o(1)$ whenever H is a subgraph of $K_{2,2,2}$ (which is the 2-blow-up of K_3). More generally, an F -counting lemma needs $t(H, \Gamma / \|\Gamma\|_1) = 1 + o(1)$ whenever H is a subgraph of the 2-blow-up of F . One might ask whether this result can be extended to L^p upper regular graphs. This is an interesting and non-trivial problem, and we leave it open for future work.

ACKNOWLEDGMENTS

We thank Oliver Riordan for suggesting the topic of Appendix C, Svante Janson for providing valuable feedback on our manuscript, Omer Tamuz for helpful discussions, Remco van der Hofstad for comments about U -statistics that led us to [20], Donald Cohn for advice about measure theory, and Patrick Wolfe and Sofia Olhede for bringing [37] to our attention.

APPENDIX A. L^p UPPER REGULARITY IMPLIES UNBOUNDED AVERAGE DEGREE

Proposition A.1. *Let $C > 0$ and $p > 1$, and let $(G_n)_{n \geq 0}$ be a C -upper L^p regular sequence of simple graphs. Then $|E(G_n)| / |V(G_n)| \rightarrow \infty$ as $n \rightarrow \infty$.*

This proposition follows immediately from the following lemma.

Lemma A.2. *For every $C > 0$ and $p > 1$ there exist $\eta_0 > 0$ and $c > 0$ such that if $0 < \eta < \eta_0$ and G is a (C, η) -upper L^p regular simple graph, then $|E(G)| / |V(G)| \geq c\eta^{-1+1/p}$.*

Proof. Let $\eta_0 = \min((2C)^{-p/(p-1)}/2, 1/3)$, and suppose G is a (C, η) -upper L^p regular simple graph with $0 < \eta < \eta_0$. We will omit all floor and ceiling signs below in order to keep the notation clean.

Let $V = V(G)$, $n = |V|$, and $m = |E(G)|$, let T be a maximal matching (a maximal set of vertex-disjoint edges) in G consisting of t edges, and let A be the set of vertices in T . We begin by showing that our choice of η_0 ensures $t \geq \eta_0 n$.

The proof of $t \geq \eta_0 n$ will amount to applying the definition (2.1) of (C, η) -upper regularity to the partition $\{A, V \setminus A\}$. To do so, we need both $|A|$ and $|V \setminus A|$ to be at least ηn . If A is too small, then we simply enlarge it to have size ηn ; we will see below that this case never actually occurs. We need not worry about the case when A is too large, because then $t \geq \eta_0 n$ automatically holds (since in that case $\eta_0 \leq 1/3$ implies $t = |A|/2 \geq (1 - \eta)n/2 \geq \eta_0 n$).

Now we can apply upper regularity. Every edge of G has a vertex in A due to the maximality of T , and so from the partition $\{A, V \setminus A\}$ and the (C, η) -upper L^p

regularity of G we obtain

$$\begin{aligned}
C^p &\geq \frac{|A|^2}{|V|^2} \left(\frac{\rho_G(A, A)}{\|G\|_1} \right)^p + \frac{|A||V \setminus A|}{|V|^2} \left(\frac{\rho_G(A, V \setminus A)}{\|G\|_1} \right)^p \\
&= \frac{|A|^2}{|V|^2} \left(\frac{|E(A)|}{|A|^2 \|G\|_1} \right)^p + \frac{|A||V \setminus A|}{|V|^2} \left(\frac{|E(G) \setminus E(A)|}{|A||V \setminus A| \|G\|_1} \right)^p \\
&\geq \frac{|A|}{|V|} \left(\frac{|E(G)|}{|A||V| \|G\|_1} \right)^p \\
&= \frac{|A|}{n} \left(\frac{n}{2|A|} \right)^p,
\end{aligned}$$

where the last inequality follows from Jensen's inequality and the convexity of $x \mapsto x^p$. Thus,

$$|A| \geq (2C)^{-p/(p-1)} n \geq 2\eta_0 n$$

and hence $t = |A|/2 \geq \eta_0 n$. (In particular, A cannot have been enlarged in the previous paragraph, because then $|A| = \eta n$ would contradict $|A| \geq 2\eta_0 n$.)

Let $\mathcal{P} = \{P_1, \dots, P_{1/\eta}\}$ be a partition of V into sets of size ηn (plus at most one remainder set of size between ηn and $2\eta n$) so that every edge of T lies entirely in some part of \mathcal{P} ; in other words, $T \subseteq \bigcup_i P_i \times P_i$. Then, by the definition of L^p upper regularity and the convexity of $x \mapsto x^p$,

$$\begin{aligned}
2Cm/n^2 = C \|G\|_1 &\geq \|G_{\mathcal{P}}\|_p \geq \left(\sum_{i=1}^{1/\eta} \frac{|P_i|^2}{|V|^2} \left(\frac{2|T \cap (P_i \times P_i)|}{|P_i|^2} \right)^p \right)^{1/p} \\
&\geq \left(\frac{\sum_{i=1}^{1/\eta} |P_i|^2}{|V|^2} \left(\frac{2|T|}{\sum_{i=1}^{1/\eta} |P_i|^2} \right)^p \right)^{1/p} \\
&= \frac{2t}{n^{2/p} (\sum_{i=1}^{1/\eta} |P_i|^2)^{(p-1)/p}} \\
&= \Omega \left(\frac{\eta_0 n}{n^{2/p} (\eta^{-1} (n\eta)^2)^{(p-1)/p}} \right) \\
&= \Omega(\eta_0 \eta^{-(p-1)/p} n^{-1}).
\end{aligned}$$

It follows that $m/n = \Omega_{p,C}(\eta^{-(p-1)/p})$, as desired. \square

APPENDIX B. PROOF OF A CHERNOFF BOUND

Proof of Lemma 7.1. Let $t = \ln(1 + \lambda)$. We have

$$\begin{aligned}
\mathbb{P}(X - \mathbb{E}X \geq \lambda q) &\leq \mathbb{E}[\exp(t(X - \mathbb{E}X - \lambda q))] \\
\text{(B.1)} \quad &= \prod_{i=1}^n \mathbb{E}[\exp(t(X_i - \mathbb{E}X_i - \lambda p_i))].
\end{aligned}$$

If X_i is distributed as Bernoulli(p_i), then

$$\begin{aligned}
\mathbb{E}[\exp(t(X_i - \mathbb{E}X_i - \lambda p_i))] &= (1 - p_i + p_i e^t) \exp(-tp_i(1 + \lambda)) \\
&\leq \exp(p_i(e^t - 1 - t(1 + \lambda))).
\end{aligned}$$

We have

$$e^t - 1 - t(1 + \lambda) = \lambda - (1 + \lambda) \ln(1 + \lambda) \leq \begin{cases} -\frac{1}{3}\lambda^2 & \text{if } 0 < \lambda \leq 1, \\ -\frac{1}{3}\lambda & \text{if } \lambda > 1. \end{cases}$$

On the other hand, if X_i is distributed as $-\text{Bernoulli}(p_i)$, then

$$\begin{aligned} \mathbb{E}[\exp(t(X_i - \mathbb{E}X_i - \lambda p_i))] &= (1 - p_i + p_i e^{-t}) \exp(tp_i(1 - \lambda)) \\ &\leq \exp(p_i(e^{-t} - 1 + t(1 - \lambda))) \end{aligned}$$

and

$$e^{-t} - 1 + t(1 - \lambda) = \frac{-\lambda}{1 + \lambda} + (1 - \lambda) \ln(1 + \lambda) \leq \begin{cases} -\frac{1}{2}\lambda^2 & \text{if } 0 < \lambda \leq 1, \\ -\frac{1}{2}\lambda & \text{if } \lambda > 1. \end{cases}$$

Thus in both cases,

$$\mathbb{E}[\exp(t(X_i - \mathbb{E}X_i - \lambda p_i))] \leq \begin{cases} \exp(-\frac{1}{3}\lambda^2 p_i) & \text{if } 0 < \lambda \leq 1, \\ \exp(-\frac{1}{3}\lambda p_i) & \text{if } \lambda > 1. \end{cases}$$

Using these bounds in (B.1), we find that

$$\mathbb{P}(X - \mathbb{E}X \geq \lambda q) \leq \begin{cases} \exp(-\frac{1}{3}\lambda^2 q) & \text{if } 0 < \lambda \leq 1, \\ \exp(-\frac{1}{3}\lambda q) & \text{if } \lambda > 1. \end{cases}$$

The same upper bound holds for $\mathbb{P}(X - \mathbb{E}X \leq -\lambda q)$ since it is equivalent to the previous case after negating all X_i 's. The result follows by combining the two bounds using a union bound. \square

APPENDIX C. UNIFORM UPPER REGULARITY

In the theory of martingale convergence, L^p boundedness implies L^p convergence when $p > 1$, but the same is not true for $p = 1$. Instead, L^1 convergence is characterized by uniform integrability. Oliver Riordan asked whether there is a similar characterization of convergence to L^1 graphons. In this appendix, we show that the answer is yes. Although bounding the L^1 norm itself is insufficient, more detailed tail bounds suffice. In fact, the same truncation arguments that work for $p > 1$ then extend naturally to $p = 1$.

Definition C.1. Let $K: (0, \infty) \rightarrow (0, \infty)$ be any function. A graphon W has *K -bounded tails* if for each $\varepsilon > 0$,

$$\|W 1_{|W| \geq K(\varepsilon)}\|_1 \leq \varepsilon.$$

A set S of graphons is *uniformly integrable* if there exists a function $K: (0, \infty) \rightarrow (0, \infty)$ such that all graphons in S have K -bounded tails.

Every graphon has K -bounded tails for some K , because we have assumed as part of our definition that all graphons are L^1 . For purposes of analyzing convergence, we consider a tail bound function K to be the L^1 equivalent of a bound on the L^p norm for $p > 1$. For comparison, note that for $K > 0$,

$$\|W 1_{|W| \geq K}\|_1 \leq \left\| W \left(\frac{|W|}{K} \right)^{p-1} \right\|_1 = \frac{\|W\|_p^p}{K^{p-1}},$$

which tends to zero as $K \rightarrow \infty$ as long as $p > 1$ and $\|W\|_p < \infty$.

Recall that L^1 upper regularity is vacuous, since every graphon is L^1 upper regular. To get the right notion of upper regularity, we simply replace L^1 boundedness with K -bounded tails:

Definition C.2. Let $K : (0, \infty) \rightarrow (0, \infty)$ and $\eta > 0$. A graphon W is (K, η) -upper regular if $W_{\mathcal{P}}$ has K -bounded tails for every partition \mathcal{P} of $[0, 1]$ with all parts of size at least η .

A sequence $(W_n)_{n \geq 0}$ of graphons is *uniformly upper regular* if there exist $K : (0, \infty) \rightarrow (0, \infty)$ and $\eta_0, \eta_1, \dots > 0$ such that $\lim_{n \rightarrow \infty} \eta_n = 0$ and W_n is (K, η_n) -upper regular.

We define (K, η) -upper regularity of a weighted graph G using the graphon $W^G / \|G\|_1$, except that we consider only partitions \mathcal{P} that correspond to partitions of $V(G)$ for which all the parts have weight at least $\eta \alpha_G$, and we require every vertex of G to have weight at most $\eta \alpha_G$.

Note that if a graph sequence has no dominant nodes and the corresponding graphon sequence is uniformly upper regular, then so is the graph sequence.

Uniform upper regularity is the proper L^1 analogue of L^p upper regularity, and imposing uniform integrability avoids the otherwise pathological behavior of L^1 graphons. Our results for L^p graphons with $p > 1$ then generalize straightforwardly to L^1 . In the remainder of this appendix, we state the results and describe the minor modifications required for their proofs.

We will need the following two lemmas, which are standard facts about uniform integrability and conditioning a uniformly integrable set of random variables on different σ -algebras.

Lemma C.3. *Let $K : (0, \infty) \rightarrow (0, \infty)$ be any function. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every graphon W with K -bounded tails and every subset I of $[0, 1]^2$ with Lebesgue measure $\lambda(I) \leq \delta$,*

$$\int_I |W| \leq \varepsilon.$$

Explicitly, δ can be chosen to be $\varepsilon / (2K(\varepsilon/2))$.

Proof. For each I satisfying $\lambda(I) \leq \varepsilon / (2K(\varepsilon/2))$,

$$\|W 1_I\|_1 \leq \|W 1_{|W| \leq K(\varepsilon/2)} 1_I\|_1 + \|W 1_{|W| \geq K(\varepsilon/2)}\|_1 \leq K(\varepsilon/2)\lambda(I) + \varepsilon/2 \leq \varepsilon. \quad \square$$

Lemma C.4. *Let S be a uniformly integrable set of graphons. Then*

$$\{W_{\mathcal{P}} : W \in S \text{ and } \mathcal{P} \text{ is a partition of } [0, 1]\}$$

is uniformly integrable.

Proof. Suppose $\|W\|_1 \leq C$ for all $W \in S$ (every uniformly integrable set is L^1 bounded). Let $\varepsilon > 0$, and let δ be such that $\|W 1_I\|_1 \leq \varepsilon$ whenever $W \in S$ and $\lambda(I) \leq \delta$, by Lemma C.3. We will show that if $K = C/\delta$, then $\|W_{\mathcal{P}} 1_{|W_{\mathcal{P}}| \geq K}\|_1 \leq \varepsilon$ for all $W \in S$ and \mathcal{P} .

Let W be in S and \mathcal{P} be a partition, and let I be the set on which $|W_{\mathcal{P}}| \geq K$. Then

$$K\lambda(I) \leq \|W_{\mathcal{P}}\|_1 \leq \|W\|_1 \leq C,$$

and hence $\lambda(I) \leq \delta$. It follows that $\|W 1_{|W_{\mathcal{P}}| \geq K}\|_1 \leq \varepsilon$, while $\|W_{\mathcal{P}} 1_{|W_{\mathcal{P}}| \geq K}\|_1 \leq \|W 1_{|W_{\mathcal{P}}| \geq K}\|_1$ thanks to the triangle inequality (look at each part of \mathcal{P}). Thus,

$$\|W_{\mathcal{P}} 1_{|W_{\mathcal{P}}| \geq K}\|_1 \leq \varepsilon,$$

as desired. \square

We begin with the analogue of Proposition 2.10.

Proposition C.5. *Let W_0, W_1, \dots and W be graphons such that $\delta_{\square}(W_n, W) \rightarrow 0$ as $n \rightarrow \infty$. Then the sequence $(W_n)_{n \geq 0}$ is uniformly upper regular.*

It follows immediately that the same also holds for graphs, as long as they have no dominant nodes.

Proof. Choose η_n so that $\eta_n \rightarrow 0$ and

$$\|W_n - W^{\sigma_n}\|_{\square} \leq \eta_n^3$$

for some measure-preserving bijection σ_n on $[0, 1]$. Then

$$\|(W_n)_{\mathcal{P}} - (W^{\sigma_n})_{\mathcal{P}}\|_{\infty} \leq \eta_n$$

whenever all the parts of \mathcal{P} have size at least η_n , as in Lemma 5.1. We would like to show that picking K large enough forces $\|(W_n)_{\mathcal{P}} 1_{|(W_n)_{\mathcal{P}}| \geq K}\|_1$ to be small.

We have

$$\begin{aligned} \|(W_n)_{\mathcal{P}} 1_{|(W_n)_{\mathcal{P}}| \geq K}\|_1 &\leq \|((W^{\sigma_n})_{\mathcal{P}} + \eta_n) 1_{|(W_n)_{\mathcal{P}}| \geq K}\|_1 \\ &\leq \|((W^{\sigma_n})_{\mathcal{P}} + \eta_n) 1_{|(W^{\sigma_n})_{\mathcal{P}}| \geq K - \eta_n}\|_1. \end{aligned}$$

If we take $K \geq 2\eta_n$ (which is possible because $\eta_n \rightarrow 0$ as $n \rightarrow \infty$), then we have an upper bound of

$$2 \|(W^{\sigma_n})_{\mathcal{P}} 1_{|(W^{\sigma_n})_{\mathcal{P}}| \geq K - \eta_n}\|_1,$$

which tends uniformly to zero as $K \rightarrow \infty$ by Lemma C.4. \square

The converse is also true: every uniformly upper regular sequence has a convergent subsequence (Theorem C.13). This is the analogue of Theorem 2.9, but we will have to develop machinery for the L^1 case before we can prove it.

Theorem C.6 (Weak regularity lemma). *Fix $K: (0, \infty) \rightarrow (0, \infty)$. For each $\varepsilon > 0$, there exists an N such that for every natural number $k \geq N$, every graphon W with K -bounded tails, and every equipartition \mathcal{P} of $[0, 1]$, there exists an equipartition \mathcal{Q} refining \mathcal{P} into $k|\mathcal{P}|$ parts such that*

$$\|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon.$$

Proof. We start by applying the L^2 weak regularity lemma (Lemma 3.3) to the truncation $W 1_{|W| \leq K(\varepsilon/4)}$, which has L^2 norm at most $K(\varepsilon/4)$. It follows that the theorem statement holds with the conclusion replaced by

$$\left\| W 1_{|W| \leq K(\varepsilon/4)} - (W 1_{|W| \leq K(\varepsilon/4)})_{\mathcal{Q}} \right\|_{\square} \leq \varepsilon/4.$$

Thus, for $U := W 1_{|W| \leq K(\varepsilon/4)}$ we can find a \mathcal{Q} such that

$$\|U - U_{\mathcal{Q}}\|_{\square} \leq \varepsilon/4.$$

Then

$$\|W - U_{\mathcal{Q}}\|_{\square} \leq \|W 1_{|W| \geq K(\varepsilon/4)}\|_1 + \|U - U_{\mathcal{Q}}\|_{\square} \leq \varepsilon/2,$$

from which it follows that $\|W - W_{\mathcal{Q}}\|_{\square} \leq \varepsilon$ (see the end of Remark 4.4 for this standard inequality). Thus, the same partitions that give an $\varepsilon/4$ -approximation of U give an ε -approximation of W . \square

The compactness of the L^p ball (Theorem 2.13) requires uniform integrability when $p = 1$:

Theorem C.7. *Let $(W_n)_{n \geq 0}$ be uniformly integrable sequence of graphons. Then there exists a graphon W such that*

$$\liminf_{n \rightarrow \infty} \delta_{\square}(W_n, W) = 0.$$

Proof. The proof is almost the same as that of Theorem 2.13 with $p = 1$, but it uses the martingale convergence theorem for uniformly integrable martingales [16, Theorem 5.5.6], rather than L^p martingales, and it uses Theorem C.6 for weak regularity. The only substantive difference is in verifying that the martingale U_1, U_2, \dots is uniformly integrable (using the notation from the proof). To do so, we start by observing that the graphons $W_{n,k}$ are uniformly integrable by Lemma C.4. Now uniform integrability for U_k follows straightforwardly, since $W_{n,k}$ converges pointwise to U_k as $n \rightarrow \infty$ and has only $|\mathcal{P}_k|$ parts. \square

Corollary C.8. *Every set of graphons that is uniformly integrable and closed under the cut metric is compact under that metric.*

We will also need analogues of the results of Section 4 for uniform upper regularity. The analogues of Lemmas 4.1 and 4.2 are straightforward (they use Lemma C.3 to replace Hölder's inequality):

Lemma C.9. *Let $K: (0, \infty) \rightarrow (0, \infty)$ and $\varepsilon > 0$. Then there exists a constant $\eta_0 = \eta_0(K, \varepsilon)$ such that the following holds for all $\eta \in (0, \eta_0)$: if $W: [0, 1]^2 \rightarrow \mathbb{R}$ is a (K, η) -upper regular graphon and $S, T \subseteq [0, 1]$ are measurable subsets with $\lambda(S) \leq \eta_0$, then*

$$|\langle W, 1_{S \times T} \rangle| \leq \varepsilon.$$

Lemma C.10. *Let $K: (0, \infty) \rightarrow (0, \infty)$ and $\varepsilon > 0$. Then there exists a constant $\eta_0 = \eta_0(K, \varepsilon)$ such that the following holds for all $\eta \in (0, \eta_0)$ and every (K, η) -upper regular graphon W : if $S, S', T, T' \subseteq [0, 1]$ are measurable sets satisfying $\lambda(S \triangle S'), \lambda(T \triangle T') \leq \eta_0$, then*

$$|\langle W, 1_{S \times T} - 1_{S' \times T'} \rangle| \leq \varepsilon.$$

Using these two lemmas, one can then prove the analogue of Theorem 4.3. Indeed, (4.2) and (4.3) (with $C = 1$) follow from Lemmas C.10 and C.3, leading again to a bound of the form (4.4). Once (4.4) is established, the proof then just proceeds as in the truncation argument in Case II of the proof of Theorem 4.3 by setting $U = W_{\mathcal{P}_n} 1_{|W_{\mathcal{P}_n}| \leq K(\varepsilon)}$ for some suitable ε . This leads to the following theorem:

Theorem C.11 (Weak regularity lemma for (K, η) -upper regular graphons). *Let $K: (0, \infty) \rightarrow (0, \infty)$ and $0 < \varepsilon < 1$. Then there exist constants $N = N(K, \varepsilon)$ and $\eta_0 = \eta_0(K, \varepsilon)$ such that the following holds for all $\eta \leq \eta_0$: for every (K, η) -upper regular graphon W , there exists a partition \mathcal{P} of $[0, 1]$ into at most 4^N measurable parts, each having measure at least η , so that*

$$\|W - W_{\mathcal{P}}\|_{\square} \leq \varepsilon.$$

Following the strategy leading to Remark 4.4 for graphons, and that leading to Theorem 4.5 and Remark 4.7 for graphs, one then gets the following version involving equipartitions and holding also for graphs.

Theorem C.12. *Let $K: (0, \infty) \rightarrow (0, \infty)$ and $0 < \varepsilon < 1$. Then there exist constants $N = N(K, \varepsilon)$ and $\eta_0 = \eta_0(K, \varepsilon)$ such that the following holds for all $\eta \leq \eta_0$: for every (K, η) -upper regular graphon W and each natural number $k \geq N$, there exists a equipartition \mathcal{P} of $[0, 1]$ into k parts so that*

$$\|W - W_{\mathcal{P}}\|_{\square} \leq \varepsilon.$$

The same holds for a weighted graph G with $W = W^G / \|G\|_1$, in which case we can use an equipartition of the vertex set, as in Remark 4.7.

Theorem C.12 now allows us to prove the analogue of Theorem 2.8 and 2.9.

Theorem C.13. *Every uniformly upper regular sequence of graphons or weighted graphs has a subsequence that converges to an L^1 graphon under the normalized cut metric.*

The proof is almost identical to that of Theorems 2.8 and 2.9: we use the transference theorem (Theorem C.12) to reduce to the compactness theorem (Theorem C.7).

Finally, we conclude by noting that the proofs of Propositions 5.2, A.1, and 2.16 carry over to uniform upper regularity:

Proposition C.14. *Let $(G_n)_{n \geq 0}$ be a uniformly upper regular sequence of weighted graphs with $\delta_{\square}(G_n / \|G_n\|_1, W) \rightarrow 0$ for some graphon W . Then the vertices of the graphs G_n may be ordered in such a way that $\|W^{G_n} / \|G_n\|_1 - W\|_{\square} \rightarrow 0$.*

Proposition C.15. *Let $(G_n)_{n \geq 0}$ be a uniformly upper regular sequence of simple graphs. Then $|E(G_n)| / |V(G_n)| \rightarrow \infty$ as $n \rightarrow \infty$.*

Proposition C.16. *Let W be any graphon, and let $(G_n)_{n \geq 0}$ be a sequence of simple graphs such that $\|G_n\|_1 \rightarrow 0$ and $\delta_{\square}(G_n / \|G_n\|_1, W) \rightarrow 0$. Let $G'_n = \mathbf{G}(|V(G_n)|, W, \|G_n\|_1)$. Then with probability 1, one can order the vertices of G_n and G'_n so that*

$$d_{\square} \left(\frac{G_n}{\|G_n\|_1}, \frac{G'_n}{\|G'_n\|_1} \right) \rightarrow 0.$$

The only substantive modification required for the proofs is that the L^p upper regularity and convexity arguments in the proof of Proposition A.1 must be replaced with applications of Lemma C.3.

REFERENCES

- [1] D. Aldous and R. Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), 1454–1508. arXiv:math/0603062 doi:10.1214/EJP.v12-463 MR2354165
- [2] D. Aldous and J. M. Steele, *The objective method: probabilistic combinatorial optimization and local weak convergence*, in H. Kesten, ed., *Probability on discrete structures*, 1–72, Encyclopaedia Math. Sci. **110**, Springer, Berlin, 2004. doi:10.1007/978-3-662-09444-0_1 MR2023650
- [3] N. Alon and J. H. Spencer, *The probabilistic method*, third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2008. MR2437651
- [4] I. Benjamini and O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. **6** (2001), no. 23, 13 pp. arXiv:math/0011019 doi:10.1214/EJP.v6-96 MR1873300
- [5] P. J. Bickel and A. Chen, *A nonparametric view of network models and Newman-Girvan and other modularities*, Proc. Natl. Acad. Sci. USA **106** (2009), 21068–21073. doi:10.1073/pnas.0907096106
- [6] B. Bollobás and O. Riordan, *Metrics for sparse graphs*, in S. Huczynska, J. D. Mitchell, and C. M. Roney-Dougal, eds., *Surveys in combinatorics 2009*, pages 211–287, London Math. Soc. Lecture Note Ser. **365**, Cambridge University Press, Cambridge, 2009. arXiv:0708.1919 MR2588543

- [7] C. Borgs, J. T. Chayes, H. Cohn, and Y. Zhao, *An L^p theory of sparse graph convergence II: LD convergence, quotients, and right convergence*, preprint, 2014. arXiv:1408.0744
- [8] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztegombi, *Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing*, Adv. Math. **219** (2008), 1801–1851. arXiv:math/0702004 doi:10.1016/j.aim.2008.07.008 MR2455626
- [9] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztegombi, *Convergent sequences of dense graphs II: Multiway cuts and statistical physics*, Ann. of Math. (2) **176** (2012), 151–219. doi:10.4007/annals.2012.176.1.2 MR2925382
- [10] S. Chatterjee and S. R. S. Varadhan, *The large deviation principle for the Erdős-Rényi random graph*, European J. Combin. **32** (2011), 1000–1017. arXiv:1008.1946 doi:10.1016/j.ejc.2011.03.014 MR2825532
- [11] A. Clauset, C. R. Shalizi, and M. E. J. Newman, *Power-law distributions in empirical data*, SIAM Rev. **51** (2009), 661–703. arXiv:0706.1062 doi:10.1137/070710111 MR2563829
- [12] A. Coja-Oghlan, C. Cooper, and A. Frieze, *An efficient sparse regularity concept*, SIAM J. Discrete Math. **23** (2010), 2000–2034. doi:10.1137/080730160 MR2594968
- [13] D. Conlon and J. Fox, *Bounds for graph regularity and removal lemmas*, Geom. Funct. Anal. **22** (2012), 1191–1256. arXiv:1107.4829 doi:10.1007/s00039-012-0171-x MR2989432
- [14] D. Conlon, J. Fox, and Y. Zhao, *Extremal results in sparse pseudorandom graphs*, Adv. Math. **256** (2014), 206–290. arXiv:1204.6645 doi:10.1016/j.aim.2013.12.004 MR3177293
- [15] D. Conlon, J. Fox, and Y. Zhao, *A relative Szemerédi theorem*, Geom. Funct. Anal., to appear. arXiv:1305.5440
- [16] R. Durrett, *Probability: theory and examples*, fourth edition, Cambridge University Press, Cambridge, 2010. MR2722836
- [17] H. Finner, *A generalization of Hölder’s inequality and some probability inequalities*, Ann. Probab. **20** (1992), 1893–1901. doi:10.1214/aop/1176989534 MR1188047
- [18] A. Frieze and R. Kannan, *Quick approximation to matrices and applications*, Combinatorica **19** (1999), 175–220. doi:10.1007/s004930050052 MR1723039
- [19] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. (2) **167** (2008), 481–547. doi:10.4007/annals.2008.167.481 arXiv:math/0404188 MR2415379
- [20] W. Hoeffding, *The strong law of large numbers for U -statistics*, North Carolina State University, Institute of Statistics Mimeograph Series No. 302, 1961. <http://repository.lib.ncsu.edu/dr/handle/1840.4/2128>
- [21] S. Janson, *Graphons, cut norm and distance, couplings and rearrangements*, New York Journal of Mathematics, NYJM Monographs **4**, State University of New York, University at Albany, Albany, NY, 2013. arXiv:1009.2376 MR3043217
- [22] Y. Kohayakawa, *Szemerédi’s regularity lemma for sparse graphs*, in F. Cucker and M. Shub, eds., *Foundations of computational mathematics*, pages 216–230, Springer, Berlin, 1997. doi:10.1007/978-3-642-60539-0_16 MR1661982
- [23] Y. Kohayakawa and V. Rödl, *Szemerédi’s regularity lemma and quasi-randomness*, in B. A. Reed and C. L. Sales, eds., *Recent advances in algorithms and combinatorics*, pages 289–351, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC **11**, Springer-Verlag, New York, 2003. doi:10.1007/0-387-22444-0_9 MR1952989
- [24] D. Kunszenti-Kovács, L. Lovász, and B. Szegedy, *Multigraph limits, unbounded kernels, and Banach space decorated graphs*, preprint, 2014. arXiv:1406.7846
- [25] L. Lovász, *Large networks and graph limits*, American Mathematical Society Colloquium Publications **60**, American Mathematical Society, Providence, RI, 2012. MR3012035
- [26] L. Lovász and B. Szegedy, *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), 933–957. arXiv:math/0408173 doi:10.1016/j.jctb.2006.05.002 MR2274085
- [27] L. Lovász and B. Szegedy, *Szemerédi’s lemma for the analyst*, Geom. Funct. Anal. **17** (2007), 252–270. doi:10.1007/s00039-007-0599-6 MR2306658
- [28] E. Lubetzky and Y. Zhao, *On replica symmetry of large deviations in random graphs*, Random Structures Algorithms, to appear. arXiv:1210.7013 doi:10.1002/rsa.20536
- [29] R. Lyons, *Asymptotic enumeration of spanning trees*, Combin. Probab. Comput. **14** (2005), 491–522. arXiv:math/0212165 doi:10.1017/S096354830500684X MR2160416
- [30] M. Mitzenmacher, *A brief history of generative models for power law and lognormal distributions*, Internet Math. **1** (2004), 226–251. doi:10.1080/15427951.2004.10129088 MR2077227

- [31] J. Nešetřil and P. Ossona de Mendez, *Sparsity. Graphs, structures, and algorithms*, Algorithms and Combinatorics **28**, Springer, Heidelberg, 2012. doi:10.1007/978-3-642-27875-4 MR2920058
- [32] J. Nešetřil and P. Ossona de Mendez, *A model theory approach to structural limits*, Comment. Math. Univ. Carolin. **53** (2012), 581–603. arXiv:1303.2865 MR3016428
- [33] J. Nešetřil and P. Ossona de Mendez, *A unified approach to structural limits and limits of graphs with bounded tree-depth*, preprint, 2013. arXiv:1303.6471
- [34] J. Nešetřil and P. Ossona de Mendez, *Modeling limits in hereditary classes: reduction and application to trees*, preprint, 2013. arXiv:1312.0441
- [35] A. Scott, *Szemerédi’s regularity lemma for matrices and sparse graphs*, Combin. Probab. Comput. **20** (2011), 455–466. arXiv:1010.0628 doi:10.1017/S0963548310000490 MR2784637
- [36] E. Szemerédi, *Regular partitions of graphs*, in J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, and D. Sotteau, eds., *Problèmes combinatoires et théorie des graphes*, pages 309–401, Colloque Inter. CNRS **260**, CNRS, Paris, 1978. MR0540024
- [37] P. J. Wolfe and S. C. Olhede, *Nonparametric graphon estimation*, preprint, 2013. arXiv:1309.5936

MICROSOFT RESEARCH, ONE MEMORIAL DRIVE, CAMBRIDGE, MA 02142
E-mail address: borgs@microsoft.com

MICROSOFT RESEARCH, ONE MEMORIAL DRIVE, CAMBRIDGE, MA 02142
E-mail address: jchayes@microsoft.com

MICROSOFT RESEARCH, ONE MEMORIAL DRIVE, CAMBRIDGE, MA 02142
E-mail address: cohn@microsoft.com

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE,
MA 02139
E-mail address: yufeiz@mit.edu