Inequalities

Yufei Zhao

yufeiz@mit.edu

1 Classical Theorems

Theorem 1. (**AM-GM**) *Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 2. (**Cauchy-Schwarz**) *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,*

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Theorem 3. (**Jensen**) *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, \ldots, x_n \in [a, b]$ and any nonnegative reals $\omega_1, \omega_2, \ldots, \omega_n$ with $\omega_1 + \omega_2 + \cdots + \omega_n = 1$, we have*

$$\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n).$$

If f is concave, then the inequality is flipped.

Theorem 4. (**Weighted AM-GM**) *Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have*

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \geq x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n}.$$

Theorem 5. (**Schur**) *Let x, y, z be nonnegative real numbers. For any $r > 0$, we have*

$$\sum_{\text{cyclic}} x^r (x - y)(x - z) \geq 0.$$

Definition 1. (**Majorization**) *Let $\mathbf{x} = (x_1, x_2, \ldots x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two sequences of real numbers. Then $\mathbf{x}$ is said to* majorize *$\mathbf{y}$ (denoted $\mathbf{x} \succ \mathbf{y}$) if the following conditions are satisfied*

- *$x_1 \geq x_2 \geq x_3 \cdots \geq x_n$ and $y_1 \geq y_2 \geq y_3 \cdots \geq y_n$; and*
- *$x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k$, for $k = 1, 2, \ldots, n - 1$; and*
- *$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$.*

Theorem 6. (**Muirhead**)[1] *Suppose that $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$, and $x_1, \ldots x_n$ are positive real numbers, then*

$$\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}.$$

where the symmetric sum is taken over all $n!$ permutations of $x_1, x_2, \ldots, x_n$.

Theorem 7. (**Karamata's Majorization inequality**) *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n) \succ (y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a, b]$. Then, we have*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

[1] Practical notes about Muirhead: (1) don't try to apply Muirhead when there are more than 3 variables, since mostly likely you won't succeed (and never, ever try to use Muirhead when the inequality is only cyclic but not symmetric, since it is incorrect to use Muirhead there) (2) when writing up your solution, it is probably safer to just deduce the inequality using weighted AM-GM by finding the appropriate weights, as this can always be done. The reason is that it is not always clear that Muirhead will be accepted as a quoted theorem.

Theorem 8. (**Power Mean**) *Let $x_1, \cdots, x_n > 0$. The power mean of order r is defined by*

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,, \quad M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r + \cdots + x_n{}^r}{n}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Theorem 9. (**Bernoulli**) *For all $r \geq 1$ and $x \geq -1$, we have*

$$(1+x)^r \geq 1 + rx.$$

Definition 2. (**Symmetric Means**) *For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t + x_1) \cdots (t + x_n)$ is called the i-th elementary symmetric function σ_i. This means that*

$$(t + x_1) \cdots (t + x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1} t + \sigma_n.$$

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 10. *Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have*

(1) (**Newton**) $\frac{S_i}{S_{i+1}} \geq \frac{S_{i-1}}{S_i}$,
(2) (**Maclaurin**) $S_i{}^{\frac{1}{i}} \geq S_{i+1}{}^{\frac{1}{i+1}}$.

Theorem 11. (**Rearrangement**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have*

$$\sum_{i=1}^n x_i y_i \geq \sum_{i=1}^n x_i y_{\sigma(i)} \geq \sum_{i=1}^n x_i y_{n+1-i}.$$

Theorem 12. (**Chebyshev**) *Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have*

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 13. (**Hölder**)[2] *Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i{}^q\right)^{\frac{1}{q}}.$$

More generally, let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^n \left(\sum_{i=1}^m x_{ij}\right)^{\omega_j} \geq \sum_{i=1}^m \left(\prod_{j=1}^n x_{ij}{}^{\omega_j}\right).$$

Theorem 14. (**Minkowski**)[3] *If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ and $p > 1$, then*

$$\left(\sum_{i=1}^n x_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n y_i{}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (x_i + y_i)^p\right)^{\frac{1}{p}}$$

[2] Think of this as generalized Cauchy, as you can use it for more than two sequences.
[3] Think of this as generalized triangle inequality.

2 A motivating example

In this section we discuss several common techniques in inequalities through following famous problem from IMO 2001 by Hojoo Lee:

Prove that
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1$$
for all positive real numbers a, b and c.

The official solution is short and mysterious:

Official solution: First we prove that
$$\frac{a}{\sqrt{a^2+8bc}}\geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$
or equivalently, that
$$(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2\geq a^{\frac{2}{3}}(a^2+8bc).$$
Indeed, this follows from applying AM-GM as follows:
$$(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2=(a^{\frac{4}{3}})^2+(b^{\frac{4}{3}}+c^{\frac{4}{3}})(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})\geq(a^{\frac{4}{3}})^2+2b^{\frac{2}{3}}c^{\frac{2}{3}}\cdot 4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}=a^{\frac{8}{3}}+8a^{\frac{2}{3}}bc=a^{\frac{2}{3}}(a^2+8bc).$$
We have similar inequalities for the other two terms, and so
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq\frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}+\frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}+\frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}=1.$$
$\square$

While the solution looks nice and short, it leaves us wonder how in the world could anyone come up with it. In particular, where did the exponent $\frac{4}{3}$ come from? Here we provide some motivation.

Isolated fudging. It is not unusual to compare individual terms of an inequality to expressions such as
$$\frac{a^r}{a^r+b^r+c^r}$$
because if the comparison turns out to be successful, we can finish off the problem right away. (Techniques of this form are sometimes called *isolated fudging*, meaning that the effort is focused on manipulating individual terms, as opposed to the inequality as a whole.) So, suppose we guess that it is possible to have an inequality of the form
$$\frac{a}{\sqrt{a^2+8bc}}\geq\frac{a^r}{a^r+b^r+c^r}.$$
Now how can we pick the candidates for r? Blind guess and check will probably get us nowhere. Luckily, there is a method that will give you the unique candidate for r (though with no promise that this r will work). Be prepared, this method will require some calculus.

Suppose that some r works. Let us consider the function
$$f(a,b,c)=\frac{a}{\sqrt{a^2+8bc}}-\frac{a^r}{a^r+b^r+c^r}.$$
What do we know about $f(a,b,c)$? Because of the inequality that we want, we need $f(a,b,c)\geq 0$ for all $a,b,c>0$. Also, by considering the point of equality, we see that $(1,1,1)$ must be a local minimum of f. So consider the partial derivative of f with respect to a (denoted $\partial f/\partial a$), meaning that we differentiate f with respect to a while treating the other variables as if they were constants. Since $(1,1,1)$ is a local minimum, this partial derivative when evaluated at $(1,1,1)$ must give zero. So let's do this computation. (If you know some multivariable calculus, it may be instructive to think in terms of $\nabla f=0$ at $(1,1,1)$.)

We have

$$\frac{\partial f}{\partial a} = \frac{\sqrt{a^2+8bc} - \frac{a^2}{\sqrt{a^2+8bc}}}{a^2+8bc} - \frac{ra^{r-1}(a^r+b^r+c^r) - a^r \cdot ra^{r-1}}{(a^r+b^r+c^r)^2}.$$

Evaluating at $(a,b,c) = (1,1,1)$ and setting the value to zero, we get $\frac{3-\frac{1}{3}}{9} - \frac{3r-r}{9} = 0$, which gives $r = \frac{4}{3}$.

Aha! Now we found the candidate for r, we can now go through the steps of the official solution (which were easy, given we know what r is). $\square$

Note that **while we used calculus to motivate our solution, we do not need to include any calculus in the solution!** In fact, calculus is best avoided in olympiad solutions as it is generally viewed unfavorably. However, since we are not required to provide the motivation to our proof, we do not need to worry about this issue. Amazing, isn't it?

Turning the table around. Seeing that the inequality is homogeneous (meaning that the transformation $(a,b,c) \mapsto (ka,kb,kc)$ does not change anything), it is natural to impose a constraint on it. So let us assume without the loss of generality that $abc = \frac{1}{8}$, so that we need to prove

$$\frac{a}{\sqrt{a^2+\frac{1}{a}}} + \frac{b}{\sqrt{b^2+\frac{1}{b}}} + \frac{c}{\sqrt{c^2+\frac{1}{c}}} \geq 1.$$

This does not seem any easier. Now, let us turn the table around and switch the roles of the constraint and the inequality.

Let

$$x = \frac{a}{\sqrt{a^2+\frac{1}{a}}}, \quad y = \frac{b}{\sqrt{b^2+\frac{1}{b}}}, \quad z = \frac{c}{\sqrt{c^2+\frac{1}{c}}}.$$

Note that $\frac{a}{\sqrt{a^2+\frac{1}{a}}} = \frac{1}{\sqrt{1+\frac{1}{a^3}}}$, so we can write a,b,c in terms of x,y,z:

$$a^3 = \frac{x^2}{1-x^2}, \quad b^3 = \frac{y^2}{1-y^2}, \quad c^3 = \frac{z^2}{1-z^2}.$$

So, the inequality that we wish to prove is $x+y+z \geq 1$ assuming $abc = \frac{1}{8}$. By considering the contrapositive, it suffices to prove that $abc < \frac{1}{8}$ assuming $x+y+z < 1$. That is, we need to prove that

$$\frac{x^2y^2z^2}{(1-x^2)(1-y^2)(1-z^2)} < \frac{1}{8^3}$$

given $x+y+z<1$. The square roots are now gone! The new inequality turns out to be extremely easy, as it is merely a straightforward application of AM-GM. Indeed, we have

$$1-x^2 > (x+y+z)^2 - x^2 = y^2+z^2+xy+xy+yz+yz+xz+xz \geq 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}$$

and similarly for $1-y^2$ and $1-z^2$. Setting this back into about inequality gives the desired result. $\square$

How to use Jensen. Since the inequality is homogeneous, we can assume that $a+b+c=1$. Note that the function $x \mapsto \frac{1}{\sqrt{x}}$ is convex. So we can use Jensen's inequality as follows:

$$\begin{aligned} a \cdot \frac{1}{\sqrt{a^2+8bc}} + b \cdot \frac{1}{\sqrt{b^2+8ca}} + c \cdot \frac{1}{\sqrt{c^2+8ab}} &\geq \frac{1}{\sqrt{a(a^2+8bc)+b(b^2+8ca)+c(c^2+8ab)}} \\ &= \frac{1}{\sqrt{a^3+b^3+c^3+24abc}}. \end{aligned}$$

So it remains to prove that

$$\frac{1}{\sqrt{a^3+b^3+c^3+24abc}} \geq 1$$

or equivalently,

$$a^3 + b^3 + c^3 + 24abc \le (a + b + c)^3.$$

This is again extremely easy, as AM-GM gives

$$(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a^2b + b^2a + b^2c + c^2b + c^2a + a^2c) + 6abc \ge 24abc.$$

□

3 Problems

The following problems are selected from a packet written by Thomas Mildorf, which can be found at `http://web.mit.edu/tmildorf/www/Inequalities.pdf`. Solutions can also be found there.

1. Show that for positive reals a, b, c
$$\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right) \ge 9a^2b^2c^2.$$
2. Let a, b, c be positive reals such that $abc = 1$. Prove that
$$a + b + c \le a^2 + b^2 + c^2.$$
3. Let $P(x)$ be a polynomial with positive coefficients. Prove that if
$$P\left(\frac{1}{x}\right) \ge \frac{1}{P(x)}$$
holds for $x = 1$, then it holds for all $x > 0$.
4. Show that for all positive reals a, b, c, d,
$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \ge \frac{64}{a + b + c + d}.$$
5. (USAMO 1980/5) Show that for all non-negative reals $a, b, c \le 1$,
$$\frac{a}{b + c + 1} + \frac{b}{c + a + 1} + \frac{c}{a + b + 1} + (1 - a)(1 - b)(1 - c) \le 1.$$
6. (USAMO 1977/5) If a, b, c, d, e are positive reals bounded by p and q with $0 < p \le q$, prove that
$$(a + b + c + d + e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \le 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$
and determine when equality holds.
7. Let $a, b, c,$ be non-negative reals such that $a + b + c = 1$. Prove that
$$a^3 + b^3 + c^3 + 6abc \ge \frac{1}{4}.$$
8. (IMO 1995/2) a, b, c are positive reals with $abc = 1$. Prove that
$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \ge \frac{3}{2}.$$
9. Let a, b, c be positive reals such that $abc = 1$. Show that
$$\frac{2}{(a + 1)^2 + b^2 + 1} + \frac{2}{(b + 1)^2 + c^2 + 1} + \frac{2}{(c + 1)^2 + a^2 + 1} \le 1.$$

10. (USAMO 1998/3) Let $a_0, \ldots, a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$ such that
$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$
Prove that
$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}.$$

11. (Romanian TST) Let a, b, x, y, z be positive reals. Show that
$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

12. The numbers $x_1, x_2, \ldots, x_n$ obey $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ and ${x_1}^3 + {x_2}^3 + \cdots + {x_n}^3 = 0$. Prove that
$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3}.$$

13. (Turkey) Let $n \geq 2$ be an integer, and $x_1, x_2, \ldots, x_n$ positive reals such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Determine the smallest possible value of
$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_3 + \cdots + x_n + x_1} + \cdots + \frac{x_n^5}{x_1 + \cdots + x_{n-1}}.$$

14. (Poland 1995) Let n be a positive integer. Compute the minimum value of the sum
$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n},$$
where $x_1, x_2, \ldots, x_n$ are positive reals such that
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n.$$

15. Prove that for all positive reals a, b, c, d,
$$a^4b + b^4c + c^4d + d^4a \geq abcd(a+b+c+d).$$

16. (Vietnam 1998) Let $x_1, \ldots, x_n$ be positive reals such that
$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}$$
Prove that
$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

17. (MOP 2002) Let a, b, c be positive reals. Prove that
$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3$$

18. (Iran 1996) Show that for all positive real numbers a, b, c,
$$(ab+bc+ca)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq \frac{9}{4}$$