Combinatorics

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1 Bijections

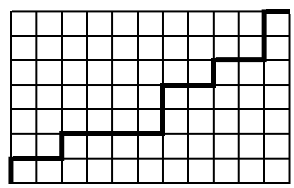
Basic examples

1. (a) Let n be a positive integer. In how many ways can one write a sum of at least two positive integers that add up to n? Consider the same set of integers written in a different order as being different. (For example, there are 3 ways to express 3 as $3 = 1 + 1 + 1 = 2 + 1 = 1 + 2$.)

   (b) Let m, n be positive integers. Determine the number of m-tuples of positive integers $(x_1, x_2, \ldots, x_m)$ satisfying $x_1 + x_2 + \cdots + x_m = n$.

   (c) Let m, n be positive integers. Determine the number of m-tuples of nonnegative integers $(x_1, x_2, \ldots, x_m)$ satisfying $x_1 + x_2 + \cdots + x_m = n$.

2. Determine the number of paths from $(0, 0)$ to (m, n) following the gridlines and moving in the up or right directions.



1.1 Catalan numbers

Let us define the nth *Catalan number* C_n by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

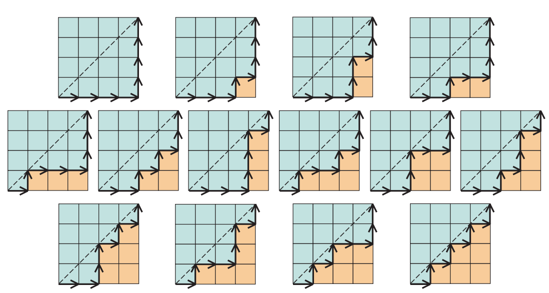
So

$$(C_0, C_1, \ldots) = (1, 1, 2, 5, 14, 42, 132, 429, \ldots).$$

There are a huge number of combinatorial interpretations of these numbers[1], and we'll discuss some of them in lecture, and leave a few more as exercise in the next section.

There are two main way of handling these problems: bijection and recurrence. The bijections often tend to be very elegant, while the recurrence method tend to be more routine. We focus on the bijection perspective but we also briefly discuss the recurrence method.

1. Show that the number of lattice paths from $(0, 0)$ to (n, n) using only up moves and right moves, and never stepping above the $x = y$ line, is C_n. E.g., for $n = 4$,

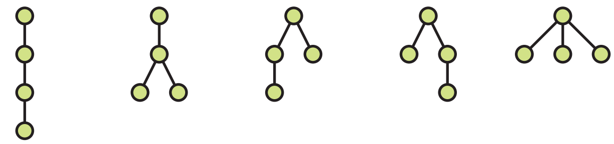


[1]See http://www-math.mit.edu/~rstan/ec/ for a list of (currently) 161 interpretations.

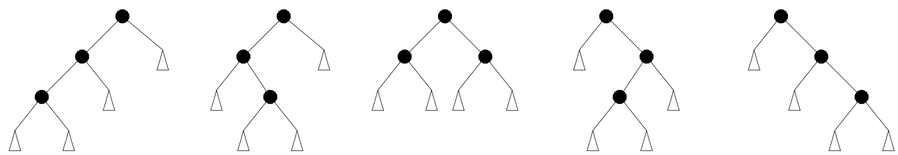
2. Show that the number of expressions containing n pairs of parentheses which are correctly matched is C_n. E.g., for $n = 3$,

$$((()))\quad (()())\quad (())()\quad ()(())\quad ()()()$$

3. Show that the number of plane trees with $n + 1$ vertices is C_n. E.g., for $n = 3$,



4. Show that the number of complete binary trees with n internal vertices is C_n. E.g., for $n = 3$,



5. Show that the number of ways that $n + 1$ factors can be completely parenthesized is C_n. E.g., for $n = 3$,

$$(((ab)c)d)\quad ((a(bc))d)\quad ((ab)(cd))\quad (a((bc)d))\quad (a(b(cd)))$$

A word about recursion. Try to show that every interpretation above gives the recurrence relation

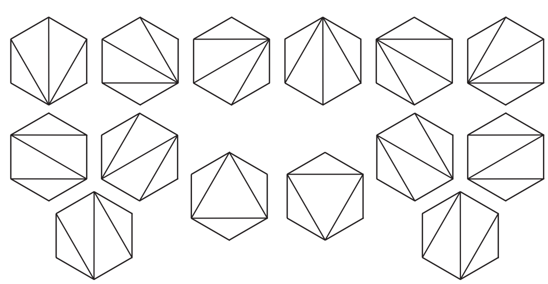
$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_1C_{n-2} + C_0C_{n-1}.$$

This is not too hard. It is mostly about how to break up a "Catalan problem" into two smaller "Catalan subproblems."

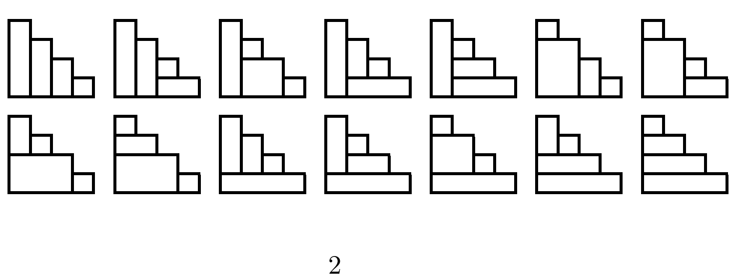
Now, if we turn the table around and ask: given recurrence relation (and the initial conditions of course), how can we arrive at the formula $C_n = \frac{1}{2n+1}\binom{2n}{n}$? This is also rather difficult unless you have seen it before. The most standard way is through generating functions (if you know generating functions, you should try to work out this computation yourself).

1.2 More Catalan exercises

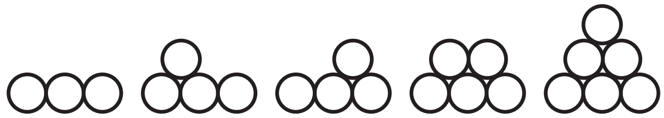
1. Show that the number of triangulations of a convex $(n + 2)$-gon is C_n. E.g., for $n = 4$,



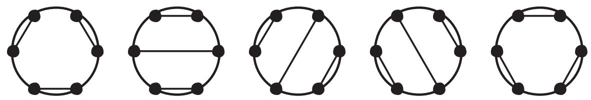
2. Show that the number of ways to tile a stairstep shape of height n with n rectangles is C_n. E.g., for $n = 4$,



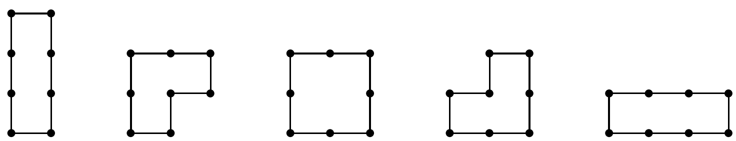
3. Show that the number of ways of stacking coins in the plane so that the bottom row consists of n consecutive coins is C_n. E.g., for $n = 3$,



4. Show that the number of ways of drawing n nonintersecting chords joining $2n$ given points on a circle is C_n. E.g., for $n = 3$,

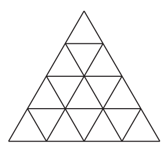


5. Show that C_n equals to the number of (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end. E.g., for $n = 3$,



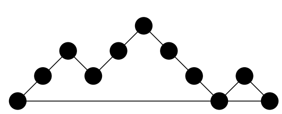
1.3 More bijections

1. A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.



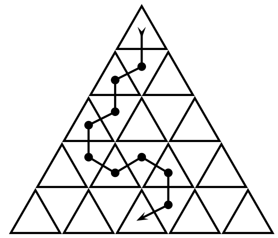
2. Form a 2000×2002 screen with unit screens. Initially, there are more than 1999×2001 unit screens which are on. In any 2×2 screen, as soon as there are 3 unit screens which are off, the 4th screen turns off automatically. Prove that the whole screen can never be totally off.

3. (Putnam 2003) A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

4. (Canada 2005) Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



5. (Putnam 2005) Let $S = \{(a, b) \mid a = 1, 2, \ldots, n, b = 1, 2, 3\}$. Determine the number of paths in S starting at $(1, 1)$ and ending at $(n, 1)$, with unit steps and passing through all the points in S exactly once each.

6. (IMO Shortlist 2002) Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

7. (USAMO 1996) An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

8. Along a one-way street there are n parking lots, and n cars numbered 1 to n enter the street in that order. Each driver of the ith car heads to his favorite parking lot $a_i \in \{1, 2, \ldots, n\}$, and, if it is free, he occupies it. Otherwise, he continues to the next free lot and occupies it. But if all succeeding lots are occupied, he leaves for good. How many sequences $(a_1, a_2, \ldots, a_n)$ are there such that every driver can park?

9. (IMO Shortlist 2002) Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfied the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k. For each n, show that there are $n!$ full sequences.

10. (IMO Shortlist 2005) In an $m \times n$ rectangular board of mn unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be colored black or white. Let N denote the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of colorings in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq M \cdot 2^{mn}$.

2 Counting in two ways

1. (IMC 2002) Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

2. (IMO 1998) In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

3. Show that if the edges of K_6, the complete graph with 6 vertices, are colored in 2 colors, then graph contains two monochromatic triangles. (Hint: count the number of monochromatic "angles")

4. (Russia 1990) There are 30 senators in a senate. Each pair of senators, the two senators are either friends of each other or enemies of each other. Every senator has exactly six enemies. Every three senators form a committee. Find the total number of committees whose members are either all friends or all enemies of each other.

5. (China 1993) Ten students ordered books. Each student ordered 3 different books. Each pair of students had ordered at least one same book. The book *Mathematics Olympiads* was the one which most (a tie being allowed) students ordered. What was the minimum number of students who ordered *Mathematics Olympiads*?

6. (USA TST 2005) Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \dots, mn\}$. Suppose that there exists a $2n$-element set T such that

   (a) each element of T is an m-element subset of S_m;

   (b) each pair of elements of T shares at most one common element; and

   (c) each element of S_m is contained in exactly two elements of T.

   Determine the maximum possible value of m in terms of n.

7. (China TST 1992) Sixteen students took part in a math competition where every problem was a multiple choice question with four choices. After the contest, it is found that any two students had at most one answer in common. Determine the maximum number of questions.

8. (IMO Shortist 2000) Let $n \geq 4$ be a fixed positive integer. Let $S = \{P_1, P_2, \dots\}$ be a set of n points in the plane such that no three are collinear and no four are concyclic. Let a_t, $1 \leq t \leq n$, denote the number of circles $P_iP_jP_k$ that contain P_t in their interiors, and let $m(S) = a_1 + a_2 + \cdots + a_n$. Prove that there exists a positive integer $f(n)$, depending only on n such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

9. Let X be a finite set with $|X| = n$, and let $A_1, A_2, \dots, A_m$ be three-element substs of X such that $|A_i \cap A_j| \leq 1$ for all $i \neq j$. Show that there exists a subset A of X with at least $\lfloor \sqrt{2n} \rfloor$ elements containing none of the A_i's.

10. (Canada 2006) Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y , Y beats Z, and Z beats X. There are no ties.

    (a) Determine the minimum number of cycle triplets possible.

    (b) Determine the maximum number of cycle triplets possible.

3 Binomial sums exercises

Prove the following identities through combinatorial interpretations. (You can assume that the variables are nonnegative integers and that all the expressions make sense.)

1. Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

2. Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

3. Prove that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

4. Prove that $1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}$.
5. Prove that $\binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots = \binom{n+1}{k+1}$.
6. Prove that $\sum_{k=0}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.
7. (Vandermonde's identity) Prove that $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}$.
8. Prove that $\sum_{i=0}^{k} (-1)^i \binom{n}{i} = (-1)^k \binom{n-1}{k}$.
9. Prove that $\sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{m-k} = 2^m \binom{n}{m}$.
10. (China 1994) Prove that $\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \binom{2n+1}{n}$.