Three Lemmas in Geometry

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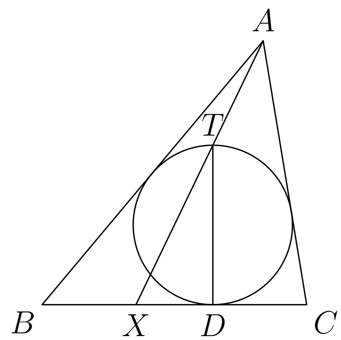
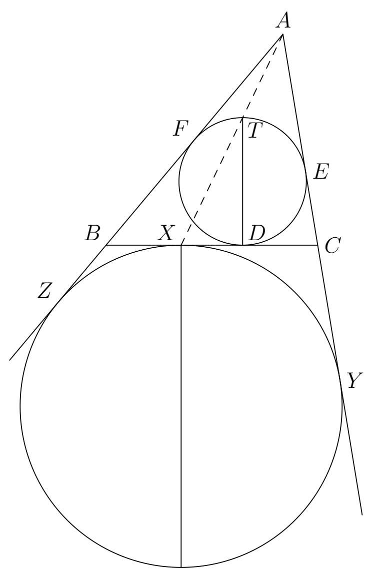
1 Diameter of incircle



Lemma 1. Let the incircle of triangle ABC touch side BC at D, and let DT be a diameter of the circle. If line AT meets BC at X, then $BD = CX$.



Proof. Assume wlog that $AB \geq AC$. Consider the dilation with center A that carries the incircle to an excircle. The line segment DT is the diameter of the incircle that is perpendicular to BC, and therefore its image under the dilation must be the diameter of the excircle that is perpendicular to BC. It follows that T must get mapped to the point of tangency between the excircle and BC. In addition, the image of T must lie on the line AT, and hence T gets mapped to X. Thus, the excircle is tangent to BC at X.

It remains to prove that $BD = CX$. Let the incircle of ABC touch sides AB and AC at F and E, respectively. Let the excircle of ABC opposite to A touch rays AB and AC at Z and Y, respectively, then using equal tangents, we have

$$\begin{aligned} 2BD &= BF + BX + XD = BF + BZ + XD = FZ + XD \\ &= EY + XD = EC + CY + XD = DC + XC + XD = 2CX. \end{aligned}$$

Thus $BD = CX$. □

Problems

1. (IMO 1992) In the plane let $\mathcal{C}$ be a circle, ℓ a line tangent to the circle $\mathcal{C}$, and M a point on ℓ. Find the locus of all points P with the following property: there exists two points Q, R on ℓ such that M is the midpoint of QR and $\mathcal{C}$ is the inscribed circle of triangle PQR.

2. (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

3. (IMO Shortlist 2005) In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

4. (Nagel line) Let ABC be a triangle. Let the excircle of ABC opposite to A touch side BC at D. Similarly define E on AC and F on AB. Then AD, BE, CF concur (why?) at a point N known as the *Nagel point*.

   Let G be the centroid of ABC and I the incenter of ABC. Show that I, G, N lie in that order on a line (known as the *Nagel line*, and $GN = 2IG$.

5. (USAMO 2001) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

6. (Tournament of Towns 2003 Fall) Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be the point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK.

7. (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

   (Hint: show that $AB + AD = CB + CD$. What does this say about the lengths along AC?)

2 Center of spiral similarity

A *spiral similarity*[1] about a point O (known as the center of the spiral similarity) is a composition of a rotation and a dilation, both centered at O. (See diagram)

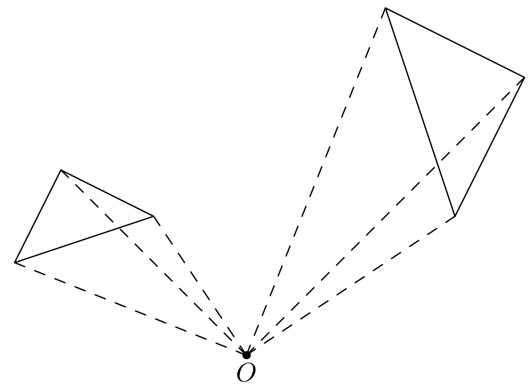


Figure 1: An example of a spiral similarity.

For instance, in the complex plane, if $O = 0$, then spiral similarities are described by multiplication by a nonzero complex number. That is, spiral similarities have the form $z \mapsto \alpha z$, where $\alpha \in \mathbb{C} \setminus \{0\}$. Here $|\alpha|$ is the dilation factor, and $\arg \alpha$ is the angle of rotation. It is easy to deduce from here that if the center of the spiral similarity is some other point, say z_0, then the transformation is given by $z \mapsto z_0 + \alpha(z - z_0)$ (why?).

Fact. Let A, B, C, D be four distinct point in the plane such that $ABCD$ is not a parallelogram. Then there exists a unique spiral similarity that sends A to B, and C to D.

Proof. Let a, b, c, d be the corresponding complex numbers for the points A, B, C, D. We know that a spiral similarity has the form $\mathbf{T}(z) = z_0 + \alpha(z - z_0)$, where z_0 is the center of the spiral similarity, and α is data on the rotation and dilation. So we would like to find α and z_0 such that $\mathbf{T}(a) = c$ and $\mathbf{T}(b) = d$. This amount to solving the system

$$z_0 + \alpha(a - z_0) = c, \qquad z_0 + \alpha(b - z_0) = d.$$

Solving it, we see that the unique solution is

$$\alpha = \frac{c - d}{a - b}, \qquad z_0 = \frac{ad - bc}{a - b - c + d}.$$

Since $ABCD$ is not a parallelogram, $a - b - c + d \neq 0$, so that this is the unique solution to the system. Hence there exists a unique spiral similarity that carries A to B and C to D. □

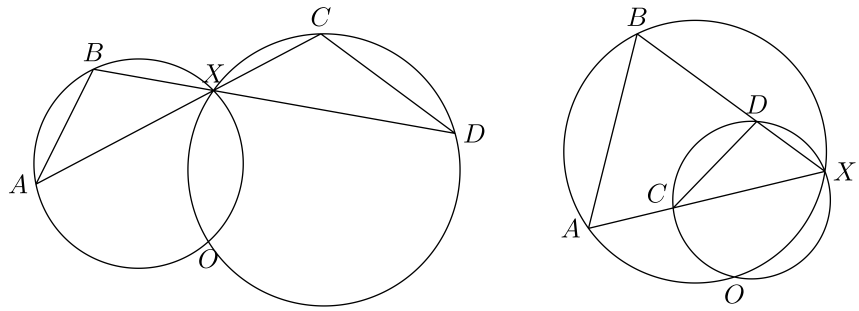
Exercise. How can you quickly determine the value of α in the above proof without even needing to set up the system of equations?

Exercise. Give a geometric argument why the spiral similarity, if it exists, must be unique. (Hint: suppose that $\mathbf{T}_1$ and $\mathbf{T}_2$ are two such spiral similarities, then what can you say about $\mathbf{T}_1 \circ \mathbf{T}_2^{-1}$?)

So we know that a spiral similarity exists, but where is its center? The following lemma tells us how to locate it.

[1] If you want to impress your friends with your mathematical vocabulary, a spiral similarity is sometimes called a *similitude*, and a dilation is sometimes called a *homothety*. (Actually, they are not quite exactly the same thing, but shhh!)

Lemma 2. Let A, B, C, D be four distinct point in the plane, such that AC is not parallel to BD. Let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the unique spiral similarity that carries A to C and B to D.



Proof. We use directed angles mod π (i.e., directed angles between *lines*, as opposed to rays) in order to produce a single proof that works in all configurations. Let $\angle(\ell_1, \ell_2)$ denote the angle of rotation that takes line ℓ_1 to ℓ_2. A useful fact is that points P, Q, R, S are concyclic if and only if $\angle(PQ, QR) = \angle(PS, SR)$.

We have

$$\angle(OA, AC) = \angle(OA, AX) = \angle(OB, BX) = \angle(OB, BD),$$

and

$$\angle(OC, CA) = \angle(OC, CX) = \angle(OD, DX) = \angle(OD, DB).$$

It follows that triangles AOC and BOD are similar and have the same orientation. Therefore, the spiral similarity centered at O that carries A to C must also carry B to D. □

Finally, it is worth mentioning that spiral similarities come in pairs. If we can send AB to CD, then we can just as easily send AC to BD using the same center.

Fact. If O is the center of the spiral similarity that sends A to C and B to D, then O is also the center of the spiral similarity that sends A to B and C to D.

Proof. Since spiral similarity preserves angles at O, we have $\angle AOB = \angle COD$. Also, the dilation ratio of the first spiral similarity is $OC/OA = OD/OB$. So the rotation about O with angle $\angle AOB = \angle COD$ followed by a dilation with ratio $OB/OA = OD/OC$ sends A to B, and C to D, as desired. □

Problems

1. (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle CBA = \angle DCA = \angle EDA.$$

   Diagonals BD and CE meet at P. Prove that line AP bisects side CD.

2. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

3. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.

4. Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.

5. (Miquel point of a quadrilateral) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let $\mathcal{C}_{ijk}$ denote the circumcircle of the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k (these circles are called *Miquel circles*). Then $\mathcal{C}_{123}, \mathcal{C}_{124}, \mathcal{C}_{134}, \mathcal{C}_{234}$ pass through a common point (called the *Miquel point*).

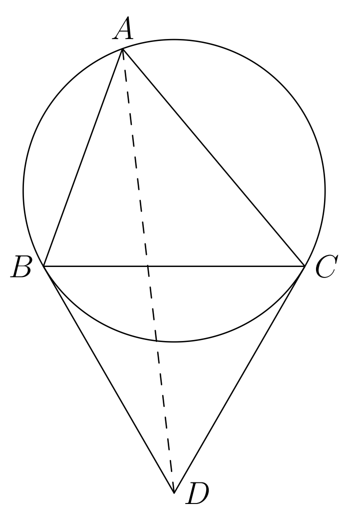
   (It's not too hard to prove this result using angle chasing, but can you see why it's almost an immediate consequence of the lemma?)

6. (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

7. (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

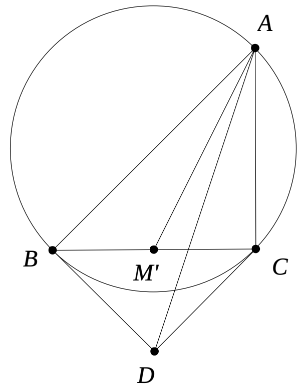
3 Symmedian

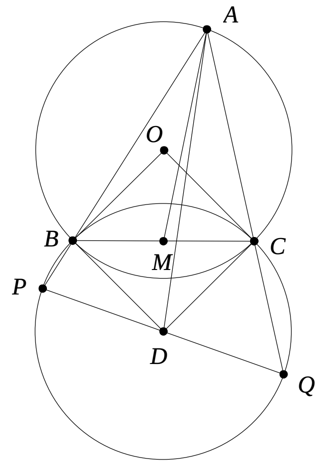
Let ABC be a triangle. Let M be the midpoint of BC, so that AM is a *median* of ABC. Let N be a point on side BC so that $\angle BAM = \angle CAN$. Then AN is a *symmedian* of ABC. In other words, *the symmedian is the reflection of the median across the angle bisector.* The following lemma gives an important property of the symmedian.

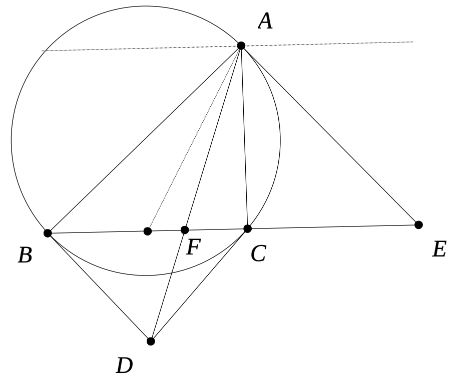


Lemma 3. Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with a symmedian of $\triangle ABC$.

We give three proofs. (The three diagram each correspond to a separate proof.) The first proof is a "sine law chase."







First proof. Let the reflection of AD across the angle bisector of $\angle BAC$ meet BC at M'. Then

$$\begin{aligned}
\frac{BM'}{M'C} &= \frac{AM'\frac{\sin\angle BAM'}{\sin\angle ABC}}{AM'\frac{\sin\angle CAM'}{\sin\angle ACB}} && \text{[Sine law on } ABM' \text{ and } ACM'] \\
&= \frac{\sin\angle BAM'}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle CAM'} && \text{[Using the tangent-chord angles]} \\
&= \frac{\sin\angle CAD}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle BAD} && \text{[From construction of } M'] \\
&= \frac{CD}{AD}\frac{AD}{BD} && \text{[Sine law on } ACD \text{ and } ABD] \\
&= 1.
\end{aligned}$$

Therefore, AM' is the median, and thus AD is the symmedian. □

Remark. Some people like to start this proof by setting M to be the midpoint of BC, and then using sine law to show that $\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\sin\angle CAM}{\sin\angle BAM}$. *I do not recommend this variation*, since it's not immediately clear that $\angle CAM = \angle BAD$ follows, especially when $\angle BAD$ is obtuse.

Next we give a synthetic proof that highlights some additional features in the configuration.

Second proof. Let O be the circumcenter of ABC and let ω be the circle centered at D with radius DB. Let lines AB and AC meet ω at P and Q, respectively. Since $\angle ABC = \angle AQP$, triangles ABC and AQP are similar. The idea is use the fact that, up to dilation, triangles ABC and AQP are reflections of each other across the angle bisector of $\angle A$.

Since

$$\angle PBQ = \angle BQC + \angle BAC = \frac{1}{2}(\angle BDC + \angle BOC) = 90°,$$

we see that PQ is a diameter of ω and hence passes through D. Let M be the midpoint of BC. Since D is the midpoint of QP, the similarity implies that $\angle BAM = \angle QAD$, from which the result follows. □

The third proof uses facts from projective geometry. Feel free to skip it if you are not comfortable with projective geometry.

Third proof. Let the tangent of Γ at A meet line BC at E. Then E is the pole of AD (since the polar of A is AE and the pole of D is BC). Let BC meet AD at F. Then point B, C, E, F are harmonic. This means that line AB, AC, AE, AF are harmonic. Consider the reflections of the four line across the angle bisector of $\angle BAC$. Their images must be harmonic too. It's easy to check that AE maps onto a line parallel to BC. Since BC must meet these four lines at harmonic points, it follows that the reflection of AF must pass through the midpoint of BC. Therefore, AF is a symmedian. □

Problems

1. (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.

2. (IMO Shortlist 2003) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

3. (Vietnam TST 2001) In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across the line PQ. Prove that the points A, S, and H are collinear.

4. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

5. Let ABC be a triangle. Let X be the center of spiral similarity that takes B to A and A to C. Show that AX coincides with a symmedian of ABC.

6. (USA TST 2008) Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

7. (USA 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

8. Let A be one of the intersection points of circles ω_1, ω_2 with centers O_1, O_2. The line ℓ is tangent to ω_1, ω_2 at B, C respectively. Let O_3 be the circumcenter of triangle ABC. Let D be a point such that A is the midpoint of O_3D. Let M be the midpoint of O_1O_2. Prove that $\angle O_1DM = \angle O_2DA$.

   (Hint: use Problem 5.)