3 Lemmas in Geometry

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1 Diameter of incircle

Lemma 1. Let the incircle of triangle $ABC$ touch side $BC$ at $D$, and let $DT$ be a diameter of the circle. If line $AT$ meets $BC$ at $X$, then $BD = CX$.

Proof. Assume wlog that $AB \geq AC$. Consider the dilation with center $A$ that carries the incircle to an excircle. The line segment $DT$ is the diameter of the incircle that is perpendicular to $BC$, and therefore its image under the dilation must be the diameter of the excircle that is perpendicular to $BC$. It follows that $T$ must get mapped to the point of tangency between the excircle and $BC$. In addition, the image of $T$ must lie on the line $AT$, and hence $T$ gets mapped to $X$. Thus, the excircle is tangent to $BC$ at $X$.

It remains to prove that $BD = CX$. Let the incircle of $ABC$ touch sides $AB$ and $AC$ at $F$ and $E$, respectively. Let the excircle of $ABC$ opposite to $A$ touch rays $AB$ and $AC$ at $Z$ and $Y$, respectively,
then using equal tangents, we have

\[ 2BD = BF + BX + XD = BF + BZ + XD = FZ + XD = EY + XD = EC + CY + XD = DC + XC + XD = 2CX. \]

Thus \( BD = CX \).

\[ \square \]

Problems

1. (IMO 1992) In the plane let \( C \) be a circle, \( \ell \) a line tangent to the circle \( C \), and \( M \) a point on \( \ell \). Find the locus of all points \( P \) with the following property: there exists two points \( Q, R \) on \( \ell \) such that \( M \) is the midpoint of \( QR \) and \( C \) is the inscribed circle of triangle \( PQR \).

2. (USAMO 1999) Let \( ABCD \) be an isosceles trapezoid with \( AB \parallel CD \). The inscribed circle \( \omega \) of triangle \( BCD \) meets \( CD \) at \( E \). Let \( F \) be a point on the (internal) angle bisector of \( \angle DAC \) such that \( EF \perp CD \). Let the circumscribed circle of triangle \( ACF \) meet line \( CD \) at \( C \) and \( G \). Prove that the triangle \( AFG \) is isosceles.

3. (IMO Shortlist 2005) In a triangle \( ABC \) satisfying \( AB + BC = 3AC \) the incircle has centre \( I \) and touches the sides \( AB \) and \( BC \) at \( D \) and \( E \), respectively. Let \( K \) and \( L \) be the symmetric points of \( D \) and \( E \) with respect to \( I \). Prove that the quadrilateral \( ACKL \) is cyclic.

4. (Nagel line) Let \( ABC \) be a triangle. Let the excircle of \( ABC \) opposite to \( A \) touch side \( BC \) at \( D \). Similarly define \( E \) on \( AC \) and \( F \) on \( AB \). Then \( AD, BE, CF \) concur (why?) at a point \( N \) known as the Nagel point.

   Let \( G \) be the centroid of \( ABC \) and \( I \) the incenter of \( ABC \). Show that \( I, G, N \) lie in that order on a line (known as the Nagel line, and \( GN = 2IG \).

5. (USAMO 2001) Let \( ABC \) be a triangle and let \( \omega \) be its incircle. Denote by \( D_1 \) and \( E_1 \) the points where \( \omega \) is tangent to sides \( BC \) and \( AC \), respectively. Denote by \( D_2 \) and \( E_2 \) the points on sides \( BC \) and \( AC \), respectively, such that \( CD_2 = BD_1 \) and \( CE_2 = AE_1 \), and denote by \( P \) the point of intersection of segments \( AD_2 \) and \( BE_2 \). Circle \( \omega \) intersects segment \( AD_2 \) at two points, the closer of which to the vertex \( A \) is denoted by \( Q \). Prove that \( AQ = D_2P \).

6. (Tournament of Towns 2003 Fall) Triangle \( ABC \) has orthocenter \( H \), incenter \( I \) and circumcenter \( O \). Let \( K \) be the point where the incircle touches \( BC \). If \( IO \) is parallel to \( BC \), then prove that \( AO \) is parallel to \( HK \).

7. (IMO 2008) Let \( ABCD \) be a convex quadrilateral with \( |BA| \neq |BC| \). Denote the incircles of triangles \( ABC \) and \( ADC \) by \( \omega_1 \) and \( \omega_2 \) respectively. Suppose that there exists a circle \( \omega \) tangent to the ray \( BA \) beyond \( A \) and to the ray \( BC \) beyond \( C \), which is also tangent to the lines \( AD \) and \( CD \). Prove that the common external tangents of \( \omega_1 \) and \( \omega_2 \) intersect on \( \omega \).

   (Hint: show that \( AB + AD = CB + CD \). What does this say about the lengths along \( AC \)?)
2 Center of spiral similarity

A *spiral similarity*\(^1\) about a point \(O\) (known as the center of the spiral similarity) is a composition of a rotation and a dilation, both centered at \(O\). (See diagram)

![Figure 1: An example of a spiral similarity.](image)

For instance, in the complex plane, if \(O = 0\), then spiral similarities are described by multiplication by a nonzero complex number. That is, spiral similarities have the form \(z \mapsto \alpha z\), where \(\alpha \in \mathbb{C} \setminus \{0\}\). Here \(|\alpha|\) is the dilation factor, and \(\arg \alpha\) is the angle of rotation. It is easy to deduce from here that if the center of the spiral similarity is some other point, say \(z_0\), then the transformation is given by \(z \mapsto z_0 + \alpha(z - z_0)\) (why?).

**Fact.** Let \(A, B, C, D\) be four distinct point in the plane such that \(ABCD\) is not a parallelogram. Then there exists a unique spiral similarity that sends \(A\) to \(B\), and \(C\) to \(D\).

**Proof.** Let \(a, b, c, d\) be the corresponding complex numbers for the points \(A, B, C, D\). We know that a spiral similarity has the form \(T(z) = z_0 + \alpha(z - z_0)\), where \(z_0\) is the center of the spiral similarity, and \(\alpha\) is data on the rotation and dilation. So we would like to find \(\alpha\) and \(z_0\) such that \(T(a) = c\) and \(T(b) = d\). This amount to solving the system

\[
z_0 + \alpha(a - z_0) = c, \quad z_0 + \alpha(b - z_0) = d.
\]

Solving it, we see that the unique solution is

\[
\alpha = \frac{c - d}{a - b}, \quad z_0 = \frac{ad - bc}{a - b - c + d}.
\]

Since \(ABCD\) is not a parallelogram, \(a - b - c + d \neq 0\), so that this is the unique solution to the system. Hence there exists a unique spiral similarity that carries \(A\) to \(B\) and \(C\) to \(D\).

**Exercise.** How can you quickly determine the value of \(\alpha\) in the above proof without even needing to set up the system of equations?

**Exercise.** Give a geometric argument why the spiral similarity, if it exists, must be unique. (Hint: suppose that \(T_1\) and \(T_2\) are two such spiral similarities, then what can you say about \(T_1 \circ T_2^{-1}\)?)

So we know that a spiral similarity exists, but where is its center? The following lemma tells us how to locate it.

\(^1\)If you want to impress your friends with your mathematical vocabulary, a spiral similarity is sometimes called a *similitude*, and a dilation is sometimes called a *homothety*. (Actually, they are not quite exactly the same thing, but shhh!)
Lemma 2. Let $A, B, C, D$ be four distinct point in the plane, such that $AC$ is not parallel to $BD$. Let lines $AC$ and $BD$ meet at $X$. Let the circumcircles of $ABX$ and $CDX$ meet again at $O$. Then $O$ is the center of the unique spiral similarity that carries $A$ to $C$ and $B$ to $D$.

Proof. We use directed angles mod $\pi$ (i.e., directed angles between lines, as opposed to rays) in order to produce a single proof that works in all configurations. Let $\angle(\ell_1, \ell_2)$ denote the angle of rotation that takes line $\ell_1$ to $\ell_2$. A useful fact is that points $P, Q, R, S$ are concyclic if and only if $\angle(PQ, QR) = \angle(PS, SR)$.

We have

$$\angle(OA, AC) = \angle(OA, AX) = \angle(OB, BX) = \angle(OB, BD),$$

and

$$\angle(OC, CA) = \angle(OC, CX) = \angle(OD, DX) = \angle(OD, DB).$$

It follows that triangles $AOC$ and $BOD$ are similar and have the same orientation. Therefore, the spiral similarity centered at $O$ that carries $A$ to $C$ must also carry $B$ to $D$.

Finally, it is worth mentioning that spiral similarities come in pairs. If we can send $AB$ to $CD$, then we can just as easily send $AC$ to $BD$ using the same center.

Fact. If $O$ is the center of the spiral similarity that sends $A$ to $C$ and $B$ to $D$, then $O$ is also the center of the spiral similarity that sends $A$ to $B$ and $C$ to $D$.

Proof. Since spiral similarity preserves angles at $O$, we have $\angle AOB = \angle COD$. Also, the dilation ratio of the first spiral similarity is $OC/OA = OD/OB$. So the rotation about $O$ with angle $\angle AOB = \angle COD$ followed by a dilation with ratio $OB/OA = OD/OC$ sends $A$ to $B$, and $C$ to $D$, as desired.

Problems

1. (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle CBA = \angle DCA = \angle EDA.$$  

Diagonals $BD$ and $CE$ meet at $P$. Prove that line $AP$ bisects side $CD$.

2. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let $E$ and $F$ be points on sides $AD$ and $BC$, respectively, such that $AE/ED = BF/FC$. Ray $FE$ meets rays $BA$ and $CD$ at $S$ and $T$, respectively. Prove that the circumcircles of triangles $SAE$, $SBF$, $TCF$, and $TDE$ pass through a common point.

3. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle $\omega$ with center $O$. Diagonals $AC$ and $BD$ meet at $P$. The circumcircles of triangles $ABP$ and $CDP$ meet at $P$ and $Q$. Assume that points $O, P,$ and $Q$ are distinct. Prove that $\angle OQP = 90^\circ$. 


4. Let $ABCD$ be a quadrilateral. Let diagonals $AC$ and $BD$ meet at $P$. Let $O_1$ and $O_2$ be the circumcenters of $APD$ and $BPC$. Let $M$, $N$ and $O$ be the midpoints of $AC$, $BD$ and $O_1O_2$. Show that $O$ is the circumcenter of $MPN$.

5. (Miquel point of a quadrilateral) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let $C_{ijk}$ denote the circumcircle of the triangle formed by the lines $\ell_i, \ell_j, \ell_k$ (these circles are called Miquel circles). Then $C_{123}, C_{124}, C_{134}, C_{234}$ pass through a common point (called the Miquel point).

(It’s not too hard to prove this result using angle chasing, but can you see why it’s almost an immediate consequence of the lemma?)

6. (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides $BC$ and $AD$ equal in length and not parallel. Let $E$ and $F$ be interior points of the sides $BC$ and $AD$ respectively such that $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, the lines $EF$ and $AC$ meet at $R$. Consider all the triangles $PQR$ as $E$ and $F$ vary. Show that the circumcircles of these triangles have a common point other than $P$.

7. (IMO Shortlist 2006) Points $A_1, B_1$ and $C_1$ are chosen on sides $BC, CA$, and $AB$ of a triangle $ABC$, respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1$, and $CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2, B_2$, and $C_2$, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points $A_3, B_3$, and $C_3$ are symmetric to $A_1, B_1, C_1$ with respect to the midpoints of sides $BC, CA$, and $AB$, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

3 Symmedian

Let $ABC$ be a triangle. Let $M$ be the midpoint of $BC$, so that $AM$ is a median of $ABC$. Let $N$ be a point on side $BC$ so that $\angle BAM = \angle CAN$. Then $AN$ is a symmedian of $ABC$. In other words, the symmedian is the reflection of the median across the angle bisector. The following lemma gives an important property of the symmedian.

**Lemma 3.** Let $ABC$ be a triangle and $\Gamma$ its circumcircle. Let the tangent to $\Gamma$ at $B$ and $C$ meet at $D$. Then $AD$ coincides with a symmedian of $\triangle ABC$.

We give three proofs. (The three diagram each correspond to a separate proof.) The first proof is a “sine law chase.”
First proof. Let the reflection of $AD$ across the angle bisector of $\angle BAC$ meet $BC$ at $M'$. Then

$$\frac{BM'}{MC} = \frac{AM'\sin \angle BAM'}{AM'\sin \angle CAM'} \frac{\sin \angle ABC}{\sin \angle ACB}$$

[Sine law on $ABM'$ and $ACM'$]

$$= \frac{\sin \angle BAM' \sin \angle ABD}{\sin \angle ACD \sin \angle CAM'}$$

[Using the tangent-chord angles]

$$= \frac{\sin \angle CAD \sin \angle ABD}{\sin \angle ACD \sin \angle BAD}$$

[From construction of $M'$]

$$= \frac{CD}{AD} \frac{AD}{BD}$$

[Sine law on $ACD$ and $ABD$]

$$= 1.$$

Therefore, $AM'$ is the median, and thus $AD$ is the symmedian. \(\square\)

Remark. Some people like to start this proof by setting $M$ to be the midpoint of $BC$, and then using sine law to show that $\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\sin \angle CAM}{\sin \angle BAM}$. I do not recommend this variation, since it’s not immediately clear that $\angle CAM = \angle BAD$ follows, especially when $\angle BAD$ is obtuse.

Next we give a synthetic proof that highlights some additional features in the configuration.

Second proof. Let $O$ be the circumcenter of $ABC$ and let $\omega$ be the circle centered at $D$ with radius $DB$. Let lines $AB$ and $AC$ meet $\omega$ at $P$ and $Q$, respectively. Since $\angle ABC = \angle AQP$, triangles $ABC$ and $AQP$ are similar. The idea is use the fact that, up to dilation, triangles $ABC$ and $AQP$ are reflections of each other across the angle bisector of $\angle A$.

Since

$$\angle PBQ = \angle BQC + \angle BAC = \frac{1}{2}(\angle BDC + \angle BOC) = 90^\circ,$$

we see that $PQ$ is a diameter of $\omega$ and hence passes through $D$. Let $M$ be the midpoint of $BC$. Since $D$ is the midpoint of $QP$, the similarity implies that $\angle BAM = \angle QAD$, from which the result follows. \(\square\)

The third proof uses facts from projective geometry. Feel free to skip it if you are not comfortable with projective geometry.

Third proof. Let the tangent of $\Gamma$ at $A$ meet line $BC$ at $E$. Then $E$ is the pole of $AD$ (since the polar of $A$ is $AE$ and the pole of $D$ is $BC$). Let $BC$ meet $AD$ at $F$. Then point $B,C,E,F$ are harmonic. This means that line $AB, AC, AE, AF$ are harmonic. Consider the reflections of the four line across the angle bisector of $\angle BAC$. Their images must be harmonic too. It’s easy to check that $AE$ maps onto a line
parallel to \( BC \). Since \( BC \) must meet these four lines at harmonic points, it follows that the reflection of \( AF \) must pass through the midpoint of \( BC \). Therefore, \( AF \) is a symmedian.

**Problems**

1. (Poland 2000) Let \( ABC \) be a triangle with \( AC = BC \), and \( P \) a point inside the triangle such that \( \angle PAB = \angle PBC \). If \( M \) is the midpoint of \( AB \), then show that \( \angle AMP + \angle BPC = 180^\circ \).

2. (IMO Shortlist 2003) Three distinct points \( A, B, C \) are fixed on a line in this order. Let \( \Gamma \) be a circle passing through \( A \) and \( C \) whose center does not lie on the line \( AC \). Denote by \( P \) the intersection of the tangents to \( \Gamma \) at \( A \) and \( C \). Suppose \( \Gamma \) meets the segment \( PB \) at \( Q \). Prove that the intersection of the bisector of \( \angle AQC \) and the line \( AC \) does not depend on the choice of \( \Gamma \).

3. (Vietnam TST 2001) In the plane, two circles intersect at \( A \) and \( B \), and a common tangent intersects the circles at \( P \) and \( Q \). Let the tangents at \( P \) and \( Q \) to the circumcircle of triangle \( APQ \) intersect at \( S \), and let \( H \) be the reflection of \( B \) across the line \( PQ \). Prove that the points \( A, S, \) and \( H \) are collinear.

4. (USA TST 2007) Triangle \( ABC \) is inscribed in circle \( \omega \). The tangent lines to \( \omega \) at \( B \) and \( C \) meet at \( T \). Point \( S \) lies on ray \( BC \) such that \( AS \perp AT \). Points \( B_1 \) and \( C_1 \) lies on ray \( ST \) (with \( C_1 \) in between \( B_1 \) and \( S \)) such that \( B_1T = BT = C_1T \). Prove that triangles \( ABC \) and \( AB_1C_1 \) are similar to each other.

5. Let \( ABC \) be a triangle. Let \( X \) be the center of spiral similarity that takes \( B \) to \( A \) and \( A \) to \( C \). Show that \( AX \) coincides with a symmedian of \( ABC \).

6. (USA TST 2008) Let \( ABC \) be a triangle with \( G \) as its centroid. Let \( P \) be a variable point on segment \( BC \). Points \( Q \) and \( R \) lie on sides \( AC \) and \( AB \) respectively, such that \( PQ \parallel AB \) and \( PR \parallel AC \). Prove that, as \( P \) varies along segment \( BC \), the circumcircle of triangle \( AQR \) passes through a fixed point \( X \) such that \( \angle BAG = \angle CAX \).

7. (USA 2008) Let \( ABC \) be an acute, scalene triangle, and let \( M, N, \) and \( P \) be the midpoints of \( BC, CA, \) and \( AB \), respectively. Let the perpendicular bisectors of \( AB \) and \( AC \) intersect ray \( AM \) in points \( D \) and \( E \) respectively, and let lines \( BD \) and \( CE \) intersect in point \( F \), inside of triangle \( ABC \). Prove that points \( A, N, F, \) and \( P \) all lie on one circle.

8. Let \( A \) be one of the intersection points of circles \( \omega_1, \omega_2 \) with centers \( O_1, O_2 \). The line \( \ell \) is tangent to \( \omega_1, \omega_2 \) at \( B, C \) respectively. Let \( O_3 \) be the circumcenter of triangle \( ABC \). Let \( D \) be a point such that \( A \) is the midpoint of \( O_3D \). Let \( M \) be the midpoint of \( O_1O_2 \). Prove that \( \angle O_1DM = \angle O_2DA \).

(Hint: use Problem 5.)