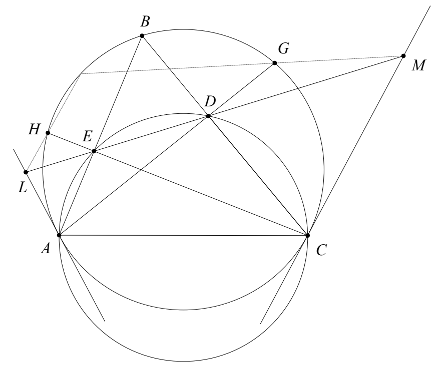
Prelude: An Unnecessary Circle

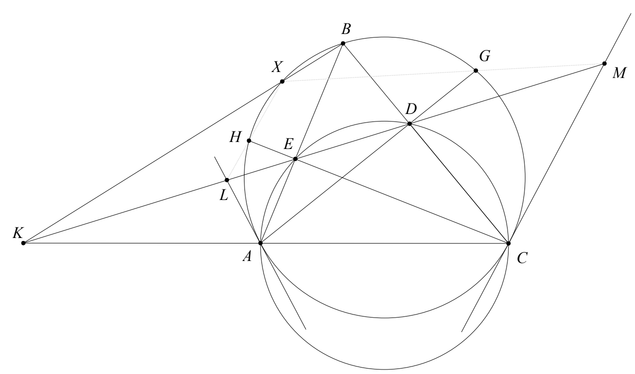
Surely you all remember the following problem from the APMO:

> (APMO 2008/3) Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.



The following solution was submitted by Chen Sun, presented below with slight modifications.

Solution. Let lines ED and AC meet at K.[1] Let line BK meet the Γ for the second time at X (if BK is tangent to Γ, then take $X = B$. Applying Pascal's theorem to the cyclic hexagon $XHCAAB$ we see that lines XH and AA (i.e. tangent to Γ at A) must meet on line KE, and so the intersection point is L. So L, H, X are collinear. Similarly, applying Pascal's theorem to $XGACCB$ shows that M, G, X are collinear. Therefore LH and MG meet at X, which is on Γ. ☐



That's it? Well, quite amazing isn't it?

While you're stunned by the simplicity of the solution, take closer look. Where was the condition that $ACDE$ is cyclic ever used?! ...Huh? What's going on?

Well, maybe the constraint was extraneous[2]. Was there some way that we could have "known" that we didn't need the cyclic condition? Can you think of way that we could have "guessed" this fact?

[1] What happens if the two lines don't meet? Well, let's not worry about that for now by pretending that we're working in a projective setting (hint hint).

[2] IMO 2003/4 anyone?