Lecture 1 : Pigeonhole Principle

Yufei Zhao

July 17, 2007

1. Let a and m be positive integers. Show that the sequence $a, a^2, a^3, a^4, \ldots$ is eventually periodic mod m.

2. Let F_n be the Fibonacci numbers, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Show that for some $n \geq 1$, F_n ends with 2007 zeros.

3. Let S be a subset of $\{1, 2, 3, \ldots, 2n\}$ with $n + 1$ elements.

   (a) Show that there are two elements in S which are relatively prime.

   (b) Show that there are two elements in S, one divisible by the other.

4. The edges of K_6, the complete graph with 6 vertices, are each colored in red or blue. Prove that there is a monochromatic triangle.

5. Let $a_1, a_2, \ldots, a_{20}$ be distinct positive integers not exceeding 70. Show that there is some k so that $a_i - a_j = k$ for four different pairs (i, j).

6. Let $a_1, a_2, \ldots, a_n$ be positive integers. Prove that we can choose some of these numbers to obtain a sum divisible by n.

7. Suppose that a and b are relatively prime integers. Show that there exist integers x and y such that $ax + by = 1$.

8. Let p and q be positive integers. Show that within any sequence of $pq+1$ distinct real numbers, there exists either a increasing subsequence of $p+1$ elements, or a decreasing subsequence of $q+1$ elements.

Problem Solving Session

July 17, 2007

1. Five lattice points are chosen in the plane lattice. Prove that you can choose two of these points such that the segment joining these points passes through another lattice point. (The plane lattice consists of all points of the plane with integral coordinates.)

2. Given 7 lines on the plane, prove that two of them form an angle less than $26°$.

3. A closed unit disc contains 7 points such that any two of them are at least unit distance apart. Show that the center of the disc is one of the 7 points.

4. At a party, certain pairs of individuals have shaken hands. Prove that there exist two persons who have shaken the same number of hands.

5. Consider the set $M = \{1, 2, 3, \ldots, 2007\}$. Prove that in any way we choose the subset X with 15 elements of M there exist two disjoint subsets A and B in X such that the sum of the members of A is equal to the sum of the members of B.

6. A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

7. Let S be a set of 6 points inside a 3×4 rectangle. Show that two of the points in S have a distance not greater than $\sqrt{5}$.

8. (Canada 2004) Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

Lecture 2 : More Pigeonholes and Some Coloring

Yufei Zhao

July 18, 2007

1. Show that there is some n for which $111\cdots111$ (with n ones) is divisible by 2007.

2. Show that there exists some integer n such that the number $n\sqrt{\pi}$ differs by at most $\frac{1}{1000000}$ from its nearest integer.

3. Let α be some irrational number. Show that for any a, b satisfying $0 < a < b < 1$, there is some positive integer n such that $a < \{n\alpha\} < b$.

4. Prove that there exists a positive integer n such that the four leftmost digits of the decimal representation of 2^n is 2007.

5. Prove that among any seven real numbers there are two, say a and b, such that $0 \le \frac{a-b}{1+ab} \le \frac{1}{\sqrt{3}}$.

6. (a) Can you tile a 6×6 board with 1×4 tiles?
   (b) Let k be a positive integer. For which positive integers m, n can a $m \times n$ rectangle can be tiled with $1 \times k$ tiles?
   (c) Let a, b, c be positive integers such that $a|b$ and $b|c$. Suppose that a rectangular box can be tiled with $a \times b \times c$ bricks. Show that one can tile the box with all the $a \times b \times c$ bricks arranged in the same orientation.

7. What is the smallest number of squares on an 8×8 chessboard which would have to be painted so that no 3×1 rectangle could be placed on the board without covering a painted square?

8. For which n is there a closed knight's tour on a $4 \times n$ chessboard?

9. (USAMO 1998) A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Determine the minimum number of mouse clicks needed to make the chessboard all one color.

Problem Solving Session

July 18, 2007

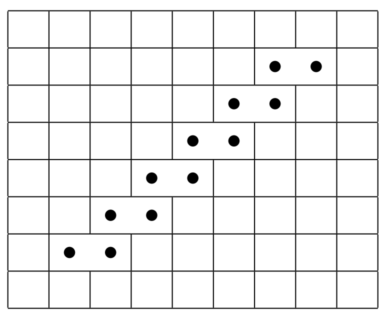
1. Show that there is a positive integer n so that $|\sin n| < 10^{-10}$.

2. Let $A_1A_2\cdots A_{2n}$ be a convex polygon. Let P be a point in the interior of the polygon, such that P does not lie on any of its diagonals. Prove that there exists a side ℓ of the polygon such that none of the lines $PA_1, PA_2, \ldots, PA_{2n}$ intersects the interior of ℓ.

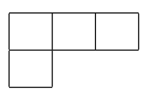
3. Let S be the set of 25 points arranged in a 5×5 unit square array. Show that among any 6 points in S, we can always find three of them so that the area of the triangle they form is at most 2.

4. Several chords of a unit circle are chosen such that no diameter intersects with more than four of them. Prove that the sum of the lengths of the chords is at most 13.

5. (Canada 2007) What is the maximum number of dominoes which can be placed on an 8×9 board if six are already placed as shown below?



6. Prove that a 4×11 rectangle cannot be tiled by L-shaped tetrominoes.



7. Show that if a rectangle can be tiled by smaller rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.

8. (IMO Shortlist 2003) Let $D_1, \ldots, D_n$ be closed discs in the plane. Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

Lecture 3 : Binomial Coefficients

Yufei Zhao

July 19, 2007

1. Show that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

2. Show that $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

3. Show that $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

4. Show that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

5. Show that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

6. Evaluate $\sum_{k=0}^{n} k^2 \binom{n}{k}$.

7. Evaluate $\binom{2007}{1} + \binom{2007}{4} + \binom{2007}{7} + \cdots + \binom{2007}{2005}$.

8. **Stirling numbers**

   (a) **Stirling numbers of the first kind**: Let $\left[{n \atop k}\right]$ denote the number of permutations on n elements with exactly k cycles. Show that $\left[{n+1 \atop k}\right] = \left[{n \atop k-1}\right] + n\left[{n \atop k}\right]$.

   (b) **Stirling numbers of the second kind**: Let $\left\{{n \atop k}\right\}$ denote the number of ways to partition $\{1, 2, \ldots, n\}$ into k non-empty sets. Show that $\left\{{n+1 \atop k}\right\} = \left\{{n \atop k-1}\right\} + k\left\{{n \atop k}\right\}$.

   (c) Show that $\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x+1)(x+2)\cdots(x+n-1)$.

   (d) Show that $\sum_{k=0}^{n} \left\{{n \atop k}\right\} x(x-1)(x-2)\cdots(x-k+1) = x^n$.

   (e) Show that $\sum_{k\geq 0} (-1)^{n-k} \left\{{m \atop k}\right\} \left[{k \atop n}\right] = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

Problem Solving Session

July 19, 2007

1. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.
2. Show that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.
3. Show that $\sum_{\text{odd } k}\binom{n}{k} = \sum_{\text{even } k}\binom{n}{k}$.
4. Evaluate $\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$.
5. Show that $\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$.
6. Prove that $\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}$.
7. Let p be a prime number, and let k be an integer such that $0 < k < p$. Show that $\binom{p}{k}$ is divisible by p.
8. (Vandermonde) Show that $\binom{a+b}{n} = \sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}$.
9. Show that $\binom{2p}{p} - 2$ is divisible by p^2 for all primes $p > 2$.
10. Let n be a nonnegative integer. Show that
$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$
11. A license plate consists of 8 digits. It is called even if it contains an even number of 0s. Find the number of even license plates.
12. Let F_n be the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that
$$\sum_{k=0}^{n}\binom{n-k+1}{k} = F_{n+2}.$$

Lecture 4 : Bijections

Yufei Zhao

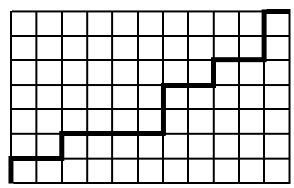
July 20, 2007

1. (a) Let n be a positive integer. In how many ways can one write a sum of at least two positive integers that add up to n? Consider the same set of integers written in a different order as being different. (For example, there are 3 ways to express 3 as $3 = 1+1+1 = 2+1 = 1+2$.)

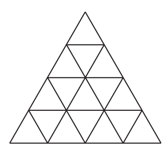
   (b) Let m, n be positive integers. Determine the number of m-tuples of positive integers $(x_1, x_2, \ldots, x_m)$ satisfying $x_1 + x_2 + \cdots + x_m = n$.

   (c) Let m, n be positive integers. Determine the number of m-tuples of nonnegative integers $(x_1, x_2, \ldots, x_m)$ satisfying $x_1 + x_2 + \cdots + x_m = n$.

2. Determine the number of paths from $(0, 0)$ to (m, n) following the gridlines and moving in the up or right directions.

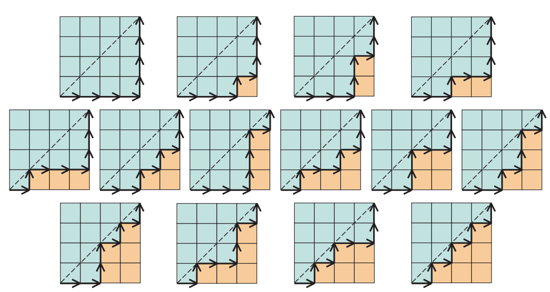


3. A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.



4. **Catalan numbers.** Let n be a positive integer and let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

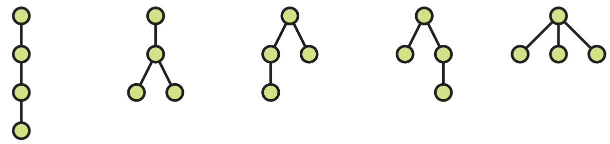
   (a) Show that the number of lattice paths from $(0, 0)$ to (n, n) using only up moves and right moves, and never stepping above the $x = y$ line, is C_n. E.g., for $n = 4$,



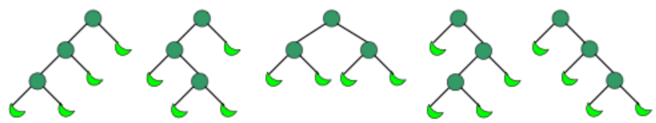
   (b) Show that the number of expressions containing n pairs of parentheses which are correctly matched is C_n. E.g., for $n = 3$,

((()))   (()())   (())()   ()(())   ()()()

(c) Show that the number of plane trees with $n+1$ vertices is C_n. E.g., for $n=3$,



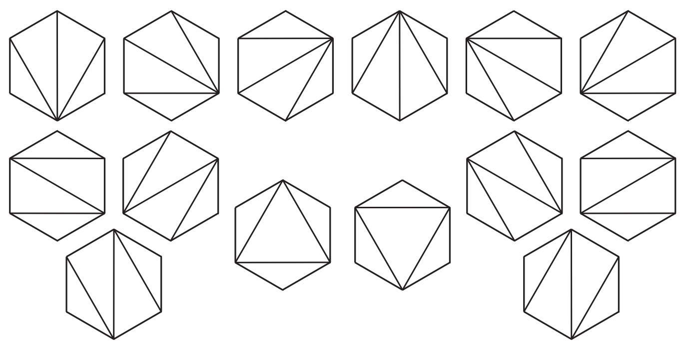
(d) Show that the number of plane binary trees with $n+1$ leaves is C_n. E.g., for $n=3$,



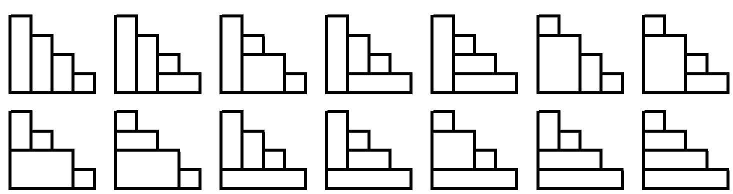
(e) Show that the number of ways that $n+1$ factors can be completely parenthesized is C_n. E.g., for $n=3$,

$$(((ab)c)d) \quad ((a(bc))d) \quad ((ab)(cd)) \quad (a((bc)d)) \quad (a(b(cd)))$$

(f) Show that the number of triangulations of a convex $(n+2)$-gon is C_n. E.g., for $n=4$,



(g) Show that the number of ways to tile a stairstep shape of height n with n rectangles is C_n. E.g., for $n=4$,



(h) Show that C_n satisfies the recurrence relation

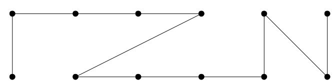
$$C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_1C_{n-2} + C_0C_{n-1}.$$

Problem Solving Session

July 20, 2007

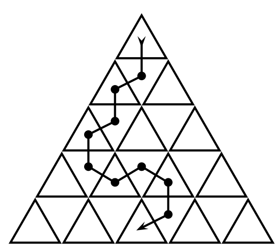
1. (AHSME 1992) Ten points are selected on the positive x-axis, $\mathbf{X}^+$, and five points are selected on the positive y-axis, $\mathbf{Y}^+$. The fifty segments connecting the ten points on $\mathbf{X}^+$ to the five points on $\mathbf{Y}^+$ are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?

2. (HMMT 2007) On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \le i \le 1, 0 \le j \le 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?

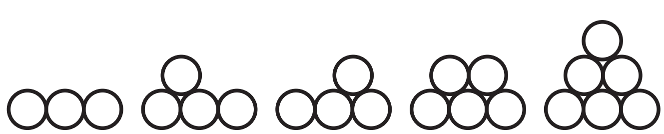


3. A triangular grid is obtained by tiling an equilateral triangle of side length n with n^2 equilateral triangles of side length 1. Determine the number of rhombi of side length 1 bounded by line segments of the grid.

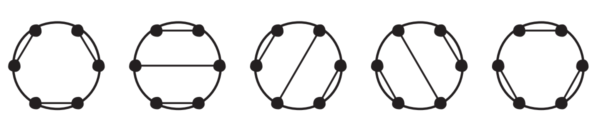
4. (Canada 2005) Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



5. Determine the number of ways of stacking coins in the plane so that the bottom row consists of n consecutive coins.



6. Determine the number of ways of drawing n nonintersecting chords joining $2n$ given points on a circle.



7. (HMMT 2007) A subset S of the nonnegative integers is called supported if it contains 0, and $k+8$, $k+9 \in S$ for all $k \in S$. How many supported sets are there?

8. (IMO Shortlist 2002) Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

Lecture 5 : More Bijections

Yufei Zhao

July 21, 2007

1. (a) Suppose that n lines are drawn on the plane, no two parallel and no three concurrent. Determine the number of regions created by these lines.

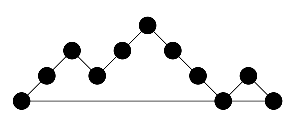
   (b) Suppose that n points are chosen on a circle and line segments are drawn connecting every pair of them, such that no three line segments are concurrent. Determine the number of regions of the disk formed by these line segments. (The sequence of answers begins as $1, 2, 4, 8, 16$.)

2. Form a 2000×2002 screen with unit screens. Initially, there are more than 1999×2001 unit screens which are on. In any 2×2 screen, as soon as there are 3 unit screens which are off, the 4th screen turns off automatically. Prove that the whole screen can never be totally off.

3. Let m, n be positive integers. Let $p(n, m)$ the number of partitions of n into m parts, and $q(n, m)$ the number of partitions of n whose largest part is m. Prove that $p(n, m) = q(n, m)$.

4. Let n be a positive integer. Let $f(n)$ denote the number of ways to partition n into distinct parts, and $g(n)$ the number of ways of partition n into odd parts. Show that $f(n) = g(n)$.

5. (Putnam 2003) A Dyck n-path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

   Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n - 1)$-paths.

Evaluation Test 1

July 21, 2007

Instructions: Read the problems carefully. Write your solutions neatly and concisely, but make sure to justify all your steps. Start each new solution on a new page and write your name and the problem number on every page.

Each problem is worth 10 points.

1. Twenty five boys and twenty five girls sit around a table. Prove that, no matter what the arrangement, it is always possible to find someone sitting in between two girls.

2. Is there a closed knight's tour on a 5×5 chessboard?

3. Evaluate
$$\binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} - \cdots + (-1)^{n+1}n^2\binom{n}{n}.$$

4. Let n be an integer greater than one, and let T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.