

18.S997 (FALL 2017) PROBLEM SET 2

1. Let the *half-graph* H_n be the bipartite graph on $2n$ vertices $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with edges $\{a_i b_j : i \leq j\}$.
 - (a) For every $\epsilon > 0$, explicitly construct an ϵ -regular partition of H_n into $O(1/\epsilon)$ parts.
 - (b) Show that there is some $c > 0$ such that for every sufficiently small $\epsilon > 0$, every integer k and sufficiently large multiple n of k , every partition of the vertices of H_n into k equal-sized parts contains at least ck pairs of parts which are not ϵ -regular.
2. Show that there is some absolute constant $C > 0$ such that for every $0 < \epsilon < 1/2$, every graph on n vertices contains an ϵ -regular pair of vertex subsets each with size at least δn , where $\delta = 2^{-\epsilon^{-C}}$.
3. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph on n vertices contains an ϵ -regular subset of vertices of size at least δn . (Here a vertex subset X is called an *ϵ -regular set* if the pair (X, X) is ϵ -regular, i.e., for all $A, B \subset X$ with $|A|, |B| \geq \epsilon|X|$, one has $|d(A, B) - d(X, X)| \leq \epsilon$.)
4. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \leq \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.
5. Show that the number of triangle-free graphs on n labeled vertices is $2^{(1/4+o(1))n^2}$.
6. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every K_4 -free graph on n vertices with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least δn .
7. For this problem you may assume either the tetrahedron¹ removal lemma for 3-uniform hypergraphs or its following corollary:
 A 3-uniform hypergraph with n vertices, where every hyperedge is contained in a unique tetrahedron, has $o(n^3)$ hyperedges.
 Deduce that if $A \subset [N]^2$ contains no axes-aligned squares (i.e., four points of the form $(x, y), (x + d, y), (x, y + d), (x + d, y + d)$, where $d \neq 0$), then $|A| = o(N^2)$.
8. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if an n -vertex graph G satisfies

$$|e(X, Y) - p|X||Y|| \leq \delta p^2 n \sqrt{|X||Y|} \quad \text{for all } X, Y \subset V(G)$$
 for some $0 < p < 1$, then the number of triangles in G is at least $(1 - \epsilon)p^3 \binom{n}{3}$.
9. Let G be an n -vertex d -regular graph. Suppose n is divisible by k . Color the vertices of G with k colors (not necessarily a proper coloring) such that each color appears exactly n/k times. Suppose that all eigenvalues, except the top one, of the adjacency matrix of G are at most d/k in absolute value. Show that there is a vertex of G whose neighborhood contains all k colors.

Problem set complete. Some hints on next page (but try the problems yourself first)

¹A tetrahedron is the set of all 3-element subsets of a 4-element vertex set—think the faces of a geometric tetrahedron

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5. You may find the following estimate helpful: $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$ for all $1 \leq k \leq n$.
6. Given an ϵ' -regular pair of vertex sets with edge-density slightly above $1/2$, find either a K_4 or a large independent set.
7. Compare to the proof in class for Szemerédi's theorem for 4-term APs. Re-parameterize the square using 4 variables so that each point of the square uses exactly 3 of the 4 variables.