PROBLEMS ON PROBABILITY

1. Three closed boxes lie on a table. One box (you don't know which) contains a \$1000 bill. The others are empty. After paying an entry fee, you play the following game with the owner of the boxes: you point to a box but do not open it; the owner then opens one of the two remaining boxes and shows you that it is empty; you may now open either the box you first pointed to or else the other unopened box, but not both. If you find the \$1000, you get to keep it. Does it make any difference which box you choose? What is a fair entry fee for this game?

2. You are dealt two cards face down from a shuffled deck of 8 cards consisting of the four queens and four kings from a standard bridge deck. The dealer looks at both of your two cards (without showing them to you) and tells you (truthfully) that at least one card is a queen. What is the probability that you have been given two queens? What is this probability if the dealer tells you instead that at least one card is a red queen? What is this probability if the dealer tells you instead that at least one card (or exactly one card) is the queen of hearts?

3. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

4. Two persons agreed to meet in a definite place between noon and one o'clock. If either person arrives while the other is not present, he or she will wait for up to 15 minutes. Calculate the probability that the meeting will occur, assuming that the arrival times are independent and uniformly distributed between noon and one o'clock.

5. Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e.

6. Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

7. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

8. Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

9. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

10. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)\,P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \le P(t) \le 1$ for $0 \le t \le 1$.

11. Choose n points $x_1, \ldots, x_n$ at random from the unit interval $[0,1]$. Let p_n be the probability that $x_i + x_{i+1} \le 1$ for all $1 \le i \le n-1$. Find a simple expression for $\sum_{n \ge 0} p_n x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$.

12. A dart, thrown at random, hits a square target. Assuming any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are integers.

13. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $1/2$. A game is finite if, with probability 1, it must end in a finite number of moves.)

14. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

15. Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists, and express this limit in the form $r(\sqrt{s} - t)$ where s and t are integers and r is a rational number.

16. The points $1, 2, \ldots, 1000$ are paired up at random to form 500 intervals $[i, j]$. What is the probability that among these intervals is one which intersects all the others?

17. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

    (i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,

    (ii) $P(y^\circ = 70^\circ)$, and

    (iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

    Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

18. In the Massachusetts MEGABUCKS lottery, six distinct integers from 1 to 36 are selected each week. Great care is exercised to insure that the selection is completely random. If $N_{\max}$ denotes the largest of the six numbers, find the expected value for $N_{\max}$.

19. (a) A fair die is tossed repeatedly. Let p_n be the probability that after some number of tosses the sum of the numbers that have appeared is n. (For instance, $p_1 = 1/6$ and $p_2 = 7/36$.) Find $\lim_{n\to\infty} p_n$.

    (b) More generally, suppose that a "die" has infinitely many faces, marked $1, 2, \ldots$. When the die is thrown, the probability is a_i that face i appears (so $\sum_{i=1}^{\infty} a_i = 1$). Let p_n be as in (a), and find $\lim_{n\to\infty} p_n$. Assume that there does not exist $k > 1$ such that if $a_i \ne 0$, then $k|i$ (otherwise it is easy to see that $\lim p_n$ doesn't exist).

20. Suppose that each of n people write down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix by rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

21. At time $t = 1$ choose two numbers x_1, y_1 uniformly and independently from $[0, 1]$. At time $t = 2$ choose two further numbers x_2, y_2, etc. What is the expected time n at which $\sum_{i=1}^{n}(x_i^2 + y_i^2) > 1$ for the first time?

    NOTE. It may seem more natural to choose just *one* number at a time, but then the answer is not as elegant.

22. A fair coin is flipped until the number of heads exceeds the number of tails. What is the expected number of flips?

PROBLEMS ON PROBABILITY GAMES

24. An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher of lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

25. A deck of cards (with 26 red cards and 26 black cards) is shuffled, and the cards are turned face up one at a time. At any point during this process before the last card is turned up, you can stay "stop." If the next card is red, you win \$1; if it is black, you win nothing. What is your best strategy? In particular, is there a strategy which gives you an expectation of better than 50 cents?

26. In the previous problem suppose that you start with \$1 and after each card is shown you can bet (at even odds) on any outcome you choose (red or black) an amount equal to any fraction of your current worth. You can certainly guarantee that you end up with \$2 — just wait until one card remains before you bet. Can you *guarantee* that you will end up with more than \$2? If so, what is the maximum amount you can be sure of winning?

27. Alice takes two slips of paper and writes a different integer on each. Bob then chooses one of the slips and looks at the integer written on it. He can then keep this slip of paper or exchange it for the other slip. If he ends up with the larger integer, he wins. Is there a strategy for Bob which gives him a probability of more than 50% of winning?

28. Suppose in the previous problem that two real numbers in the interval $[0, 1]$ are chosen uniformly at random. Alice looks at the two numbers and then decides which one to show Bob. Now if Alice chooses optimally can Bob do better than break even? What are the optimal strategies of Bob and Alice?

MORE PROBLEMS

29. Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

30. Let a, b, c, d, e, f be integers selected from the set $\{1, 2, \ldots, 100\}$, uniformly and at random with replacement. Set

$$M = a + 2b + 4c + 8d + 16e + 32f.$$

What is the expected value of the remainder when M is divided by 64?

31. The number 2019 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set

$$\{0, 1, 2, \ldots, \lceil 2.01a \rceil\}$$

uniformly at random (here $\lceil \bullet \rceil$ is the ceiling function). Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

32. Let $\mathcal{A}$ be a σ-algebra with countable cardinality. Prove that $\mathcal{A}$ is finite and determine the possible values of its cardinality.

33. Kevin has $2^n - 1$ cookies, each labeled with a unique nonempty subset of $\{1, 2, \ldots, n\}$. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie. Determine the expected value of the number of days that Kevin eats a cookie before all cookies are gone.