GENERATING FUNCTIONS

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1. Ordinary Generating Functions

A generating function is just a way of storing the data of a sequence that happens to be extremely useful.

Instead of $\{a_n\}$, we write $\sum_{n=0}^{\infty} a_n x^n$, a power series in x.

Why is this useful?

- Compute closed-form expressions
- Convert less-natural operations on sequences (e.g. convolution) to simpler ones.
- Prove combinatorial identities

Example (Classic)**.** Let $a_0 = 1$, and for $n \geq 1$, let $a_n := \sum_{i=0}^{n-1} a_i a_{n-1-i}$. Find a closed-form expression for a_i.

Solution. Let $f(x) = \sum a_n x^n$. So $xf(x)^2 + 1 = f(x)$. Solving, $f(x) = \frac{1-\sqrt{1-4x}}{2x}$.

$$\begin{aligned}
\sqrt{1-4x} &= (1-4x)^{\frac{1}{2}} \\
&= \sum_{i=0}^{\infty} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(\frac{3-2i}{2}\right)}{i!}(-1)^i 4^i x^i \\
&= \sum_{i=0}^{\infty} -2^i \frac{1\cdot 3\cdots(2i-3)}{i!} x^i \\
&= \sum_{i=0}^{\infty} -2^i \frac{(2i)!}{i!2^i i!(2i-1)} x^i \\
&= -\sum_{i=0}^{\infty} \frac{1}{2i-1}\binom{2i}{i} x^i,
\end{aligned}$$

so $\frac{1-\sqrt{1-4x}}{2x} = \sum_{i=0}^{\infty} \frac{1}{2(2i+1)}\binom{2(i+1)}{i+1} x^i$. (Shift index by 1.) We can rewrite this coefficient in the usual way as $\frac{1}{i+1}\binom{2i}{i}$.

Example (From Putnam)**.** Let $A \cup B$ be a partition of the nonnegative integers. Suppose that for every $n \geq 0$, the number of solutions to $a_1 + a_2 = n$, $a_1 \neq a_2 \in A$, is the same as the number of solutions to $b_1 + b_2 = n$, $b_1 \neq b_2 \in B$. Find all possibilities for A and B.

Solution. Let 1_A be the indicator function of A. Define $f(x) = \sum_{i=0}^{\infty} 1_A(i) x^i$, and $g(x)$ similarly for B. Then $f(x)^2$ gives the number of solutions to $a_1 + a_2 = n$. The solutions where $a_1 = a_2$ are given by $f(x^2)$.

So $f(x)^2 - f(x^2) = g(x)^2 - g(x^2)$. That is, $f(x^2) - g(x^2) = f(x)^2 - g(x)^2 = (f(x) - g(x))(f(x) + g(x)) = (f(x) - g(x))\frac{1}{1-x}$.

So $f(x) - g(x) = (f(x^2) - g(x^2))(1 - x)$. Iterating, $f(x) - g(x) = (1 - x)(1 - x^2)\cdots(1 - x^{2^{n-1}})(f(x^{2^n}) - g(x^{2^n}))$. Assuming that $0 \in A$, $f(x^{2^n}) - g(x^{2^n}) \to 1$ as $n \to \infty$. So $f(x) - g(x) = \prod_{i=0}^{\infty}(1 - x^{2^i})$.

The right hand side has $+1$ and -1 as coefficients depending on the terms in the binary expansion of n. So A and B must be, in some order, the set of nonnegative integers with an even and odd number of ones in their binary representation.

Example (USA TST 2010)**.** Let $m, n \in \mathbb{Z}^+$, $m \geq n$. Let $S_{m,n}$ be the set of all n-term sequences of positive integers $(a_1, \ldots, a_n)$ such that $a_1 + \cdots + a_n = m$. Show that

$$\sum_{S_{m,n}} 1^{a_1} 2^{a_2} \cdots n^{a_n} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m.$$

Solution. The giant sum on the left indicates that we'll likely be looking at a product of generating functions. We'd like to be able to sum over all n-tuples, not just those that sum to m, as the sum over all n-tuples of positive integers can be broken apart into several subsums.

Thus we sum the left side over all such m, first multiplying each term by x^m to form a generating function. We obtain the generating function

$$\begin{aligned}
f(x) &:= \sum_{m=0}^{\infty} \left(\sum_{S_{m,n}} 1^{a_1} 2^{a_2} \cdots n^{a_n} \right) x^m \\
&= \sum_{m=0}^{\infty} \sum_{S_{m,n}} (x)^{a_1} (2x)^{a_2} \cdots (nx)^{a_n} \\
&= \sum_{a_1, \ldots, a_n \in \mathbb{Z}^+} (x)^{a_1} (2x)^{a_2} \cdots (nx)^{a_n} \\
&= \left(\sum_{a_1=1}^{\infty} x^{a_1} \right) \left(\sum_{a_1=1}^{\infty} (2x)^{a_2} \right) \cdots \left(\sum_{a_1=1}^{\infty} (nx)^{a_n} \right) \\
&= \left(\frac{x}{1-x} \right) \left(\frac{2x}{1-2x} \right) \cdots \left(\frac{nx}{1-nx} \right) \\
&= \frac{n! x^n}{(1-x)(1-2x)\cdots(1-nx)}.
\end{aligned}$$

The corresponding generating function for the right side is

$$\begin{aligned} g(x) &:= \sum_{m=0}^{\infty} \left(\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m \right) x^m \\ &= \sum_{m=0}^{\infty} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} ((n-i)x)^m \\ &= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \sum_{m=0}^{\infty} ((n-i)x)^m \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i \binom{n}{i}}{1-(n-i)x}. \end{aligned}$$

What to do now? This is simply a case of partial fractions. We must have

$$f(x) = \frac{n!x^n}{(1-x)(1-2x)\cdots(1-nx)} = (-1)^n + \sum_{i=1}^{n} \frac{c_i}{1-ix}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$. Applying the usual trick of multiplying by $(1-ix)$ and substituting $x = \frac{1}{i}$, we see that

$$c_i = \frac{n!}{i^n \prod_{j \neq i} \left(1 - \frac{j}{i}\right)} = \frac{n!}{i \prod_{j \neq i} (i-j)}.$$

Now, $\prod_{j \neq i} (j-i) = (-1)^{n-i}(i-1)!(n-i)!$. Therefore, $c_i = (-1)^{n-i}\binom{n}{i}$. Putting this together,

$$f(x) = (-1)^n + \sum_{i=1}^{n} \frac{(-1)^{n-i}\binom{n}{i}}{1-ix}.$$

But this is the same sum we obtained for $g(x)$, just with i replaced by $n-i$ and an extra $(-1)^n$. Thus $f(x) = g(x) + (-1)^n$, so for all $m \geq 1$ the coefficient of x in f is the same as that in g, as desired.

2. Exponential Generating Functions

Instead of $\sum a_n x^n$, use $\sum \frac{a_n}{n!} x^n$.

Why is this helpful? Well, for one thing, if our sequence grows very fast, the usual g.f. might not converge.

Also, multiplication becomes, instead of convolution, $\sum a_i b_{n-i} \binom{n}{i}$. In other words, if a_n counts the number of A-structures on n objects, and b_n counts the number of B-structrues on n objects, the product of the egf's becomes the number of ways to write n objects as the union of an A-structure and a B-structure.

For example, let a_n be the number of trees on n labelled vertices, and $f_A(x)$ be the corresponding EGF. Let b_n be the number of cycles on n labelled vertices, and let $f_B(x)$ be the corresponding EGF. Then $f_A f_B$ is the EGF for the number of ways to have a graph on n labelled vertices that is the disjoint union of a tree and a cycle.

What if we wanted to instead find the number of ways to have a disjoint union of 2 trees on n vertices? We would then have $\frac{f_A^2}{2}$, where the 2 is because we can switch the two trees.

Similarly, if we wanted to have a disjoint union of k trees, we would have $\frac{f_A^k}{k!}$.

So if we wanted to find the number of forests on n vertices, it would have EGF $\sum_{k=0}^{\infty} \frac{f_A^k}{k!} = e^{f_A}$.

What we have just shown in this example is the following:

If f is the EGF for the number of A-structures on n objects, e^f is the EGF for the number of partitions into A-structures on n objects.

Example. A permutation is a partition of n elements into cycles. There are $(n-1)!$ cycles on n vertices, so the EGF for cycles is $\sum \frac{x^i}{i} = -\ln(1-x)$.

The EGF for permutations is simply $\sum x^i = \frac{1}{1-x}$.

So our identity just says that $e^{-\ln(1-x)} = \frac{1}{1-x}$, which is not illuminating but is a nice check.

Example. What about set partitions? How many ways can you partition the set $\{1, \ldots, n\}$ into other sets? Well, for each $i > 0$, there is one set of size i, so the egf for sets (of size > 0) is $\sum_{i=1}^{\infty} \frac{x^i}{i!} = e^x - 1$, so the EGF for set partitions is e^{e^x-1}.

Example. How many derangements are there on n elements? Well, these are just the permutations with no cycles of size 1. The EGF for cycles of size more than 1 is $-\ln(1-x) - x$, so we get $e^{-\ln(1-x)-x} = \frac{1}{1-x}e^{-x}$ as the EGF. This equals $\left(\sum x^i\right)\left(\sum (-1)^i \frac{x^i}{i!}\right) = \sum_{n=0}^{\infty} x^n \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

So there are $n!\left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right)$ derangements on n elements. Could use PIE, but this way requires less thought!

Example. How many cycles of length k does the average permutation of size n have?

If we weight each length-k cycles with weight t, we get that $\sum_{n,\sigma \in S_n} \frac{1}{n!} t^{\#\text{ of length k cycles}} x^n = e^{x+\frac{x^2}{2}+\cdots+t\frac{x^k}{k}+\cdots}$. Differentiating and setting $t = 1$, we get $\frac{x^k}{k} e^{-\ln(1-x)} = \frac{x^k}{k(1-x)} = \sum_{n=k}^{\infty} \frac{x^n}{k}$. So (obviously) there are no length-k cycles in permutations of length $< k$, and for all $n > k$, the average permutation has $\frac{1}{k}$ cycles of length k.

A final example, to show another way of using generating functions:

Example. Consider the set of all alternating permutations on n vertices–that is, permutations on $[n]$ with $a_1 < a_2 > a_3 < a_4 > \cdots$. Roughly how many such permutations are there?

We'll want an EGF here; there are too many alternating permutations for a regular GF to suffice.

Casework on the location of 1 yields that (for $n > 1$) $f(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2i} a_{2i} a_{n-2i-1}$. Similarly, $f(n) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n-1}{2i+1} a_{2i+1} a_{n-2i-2}$, caseworking on the location of n.

Adding, $2f(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} a_i a_{n-i-1}$.

So if $F(x) = \sum f(n)x^n$, then $2F'(x) = F(x)^2 + 1$. This differential equation has solution $f(x) = \tan\left(\frac{x}{2} + C\right)$.

Setting $x = 0$, we want $F(0) = 1$, so $C = \frac{\pi}{4}$. Therefore, $f(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \sec x + \tan x$.

One nice feature here is that sec gives the even terms, and tan gives the odd terms.

So, how fast does this sequence grow? Well, we have a pole at $\frac{\pi}{2}$, so (likely) the sequence will grow at rate $\left(\frac{2}{\pi}\right)^n n!$.