PROBLEMS ON POLYNOMIALS

NOTE. The terms "root" and "zero" of a polynomial are synonyms. In general, if the problem describes a polynomial without saying what kind of coefficients it has, you are to assume *real* coefficients.

1. (a) Determine all rational values for which a, b, c are the roots of

$$x^3 + ax^2 + bx + c = 0.$$

   (b) Show that the only real polynomials $\prod_{i=0}^{n-1}(x - a_i) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in addition to those given by (a) are $x^n, x^2 + x - 2$, and exactly two others, which are approximately equal to

$$x^3 + .56519772x^2 - 1.76929234x + .63889690$$

   and

$$x^4 + x^3 - 1.7548782x^2 - .5698401x + .3247183.$$

2. Assuming that all the roots of the cubic equation $x^3 + ax^2 + bx + c$ are real, show that the difference between the greatest and the least roots is not less than $\sqrt{a^2 - 3b}$ nor greater than $2\sqrt{(a^2 - 3b)/3}$.

3. The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z) + 1$ and $Q(z) + 1$. Prove that $P(z) = Q(z)$. (On the original Exam, the assumption that $P(z)$ and $Q(z)$ are nonconstant was inadvertently omitted.)

4. If $a_0, a_1, \dots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

   show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real root.

5. Determine all polynomials of the form

$$\sum_0^n a_i x^{n-i} \text{ with } a_i = \pm 1$$

   $(0 \le i \le n,\ 1 \le n < \infty)$ such that each has only real zeros.

6. Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2 + 1)P(x)P'(x) + x(P(x)^2 + P'(x)^2).$$

   Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n - 1$ distinct real roots.

7. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

8. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for each $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

9. Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

   (i) $p(r_i) = 0, \quad i = 1, 2, \ldots, n,$

   and

   (ii) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, 2, \ldots, n-1,$

   where $p'(x)$ denotes the derivative of $p(x)$.

10. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

11. (a) (relatively easy) Let k be the smallest positive integer with the following property:

    There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients.

    Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

    (b) (considerably more difficult) Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

12. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

13. (a) Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

(b) Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$ in the half-open interval $[0, 1)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k\to\infty} N_k = 2N.$$

14. Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \end{aligned}$$

where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

15. Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \qquad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

16. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

17. Does there exist a finite set M of nonzero real numbers, such that for any positive integer n, there exists a polynomial of degree at least n with all coefficients in M, all of whose roots are real and belong to M?

18. Suppose that $a, b, c \in \mathbb{C}$ are such that the roots of the polynomial $z^3 + az^2 + bz + c$ all satisfy $|z| = 1$. Prove that the roots of $x^3 + |a|x^2 + |b|x + |c|$ all satisfy $|x| = 1$.

19. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree n with complex coefficients a_i. Suppose that the roots of $P(x)$ are $x_1, x_2, \cdots, x_n$, i.e., we have $P(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$. The *discriminant* $\Delta(P(x))$ is defined by

$$\Delta(P(x)) = \prod_{1\leq i<j\leq n} (x_i - x_j)^2.$$

Show that

$$\Delta(x^n + ax + b) = (-1)^{\binom{n}{2}} \left(n^n b^{n-1} + (-1)^{n-1}(n-1)^{n-1}a^n\right).$$

HINT. First note that

$$P'(x) = P(x)\left(\frac{1}{x-x_1} + \cdots + \frac{1}{x-x_n}\right).$$

Use this formula to establish a connection between $\Delta(P(x))$ and the values $P'(x_i)$, $1 \leq i \leq n$.

20. Let $P_n(x) = (x+n)(x+n-1)\cdots(x+1) - (x-1)(x-2)\cdots(x-n)$. Show that all the zeros of $P_n(x)$ are purely imaginary, i.e., have real part 0.

21. Let $P(x)$ be a polynomial with complex coefficients such that every root has real part a. Let $z \in \mathbb{C}$ with $|z| = 1$. Show that every root of the polynomial $R(x) = P(x-1) - zP(x)$ has real part $a + \frac{1}{2}$.

22. Let $d \geq 1$. It is not hard to see that there exists a polynomial $A_d(x)$ of degree d such that

$$F_d(x) := \sum_{n \geq 0} n^d x^n = \frac{A_d(x)}{(1-x)^{d+1}}. \tag{9}$$

For instance, $A_1(x) = x$, $A_2(x) = x + x^2$, $A_3(x) = x + 4x^2 + x^3$. Show that every root of $A_d(x)$ is real. HINT. First obtain a recurrence for $A_d(x)$ by differentiating (9).

23. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial with complex coefficients. Choose $j \in \{0, \ldots, n\}$ so that the roots of P can be labeled $\alpha_1, \ldots, \alpha_n$ with

$$|\alpha_1|, \ldots, |\alpha_j| > 1, \qquad |\alpha_{j+1}|, \ldots, |\alpha_n| \leq 1.$$

Prove that

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + \cdots + |a_{n-1}|^2 + 1}.$$

HINT. One approach is to deduce this from an identity involving the polynomials $(z-\alpha_1)\cdots(z-\alpha_j)$ and $(\alpha_{j+1}z - 1)\cdots(\alpha_n z - 1)$.

24. Let $Q(x)$ be any monic polynomial of degree n with real coefficients. Prove that

$$\sup_{x \in [-2,2]} |Q(x)| \geq 2.$$

HINT. Let $P_n(x)$ be the monic polynomial satisfying

$$P_n(2\cos\theta) = 2\cos(n\theta) \qquad (\theta \in \mathbb{R}),$$

and examine the values of $P_n(x) - Q(x)$ at points where $|P_n(x)| = 2$.

OPTIONAL. Prove that equality only holds for $Q = P_n$.

25. Let $P(x), Q(x)$ be two polynomials with all real roots $r_1 \le r_2 \le \cdots \le r_n$ and $s_1 \le s_2 \le \cdots \le s_{n-1}$, respectively. We say that $P(x)$ and $Q(x)$ are *interlaced* if

$$r_1 \le s_1 \le r_2 \le s_2 \le \cdots \le s_{n-1} \le r_n.$$

Prove that $P(x)$ and $Q(x)$ are interlaced if and only if the polynomial $P + tQ$ has all real roots for all $t \in \mathbb{R}$.

26. Let $P(x)$ be a polynomial with real coefficients. For $t \in \mathbb{R}$, let $V(P, t)$ denote the number of sign changes in the sequence

$$P(t), P'(t), P''(t), \ldots.$$

(A *sign change* in a sequence is a pair of terms, one positive and one negative, with only zeros in between.) Prove that for any $a, b \in \mathbb{R}$, the number of roots of P in the half-open interval $(a, b]$, counted with multiplicities, is equal to $V(P, a) - V(P, b)$ minus a nonnegative even integer. Then deduce Descartes's rule of signs as a corollary.

27. Let $P(x)$ be a squarefree polynomial with real coefficients. Define the sequence of polynomials $P_0, P_1, \ldots$ by setting $P_0 = P$, $P_1 = P'$, and

$$P_{i+2} = -\text{rem}(P_i, P_{i+1}),$$

where $\text{rem}(A, B)$ means the remainder upon Euclidean division of A by B; upon arriving at a nonzero constant polynomial P_r, stop. Prove that for any $a, b \in \mathbb{R}$, the number of zeros of P in $(a, b]$ is $\sigma(a) - \sigma(b)$, where $\sigma(t)$ is the number of sign changes in the sequence

$$P_0(t), P_1(t), \ldots, P_r(t).$$

28. Let $f(x)$ be a non-constant polynomial with integer coefficients. Show that there is an integer n such that $f(n)$ is not prime.

29. Call a polynomial $P(x_1, \ldots, x_k)$ good if there exist 2×2 real matrices $A_1, \ldots, A_k$ such that

$$P(x_1, \ldots, x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.

30. How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z)| \le 2$ for any complex number z of unit length?

31. Let f be a nonzero polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \ldots$ by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \neq 0$. Prove that there exists a number N such that for every $n \ge N$, all the roots of f_n are real.

32. Find all polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ ($a_n \neq 0$) satisfying the following two conditions:

(i) $(a_0, a_1, \ldots, a_n)$ is a permutation of the numbers $(0, 1, \ldots, n)$

and

(ii) all roots of $P(x)$ are rational numbers.

33. Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$P(P(\ldots(P(x)))) = 0,$$

where there are 2010 applications of P?

34. Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.

35. Let k be a positive integer, and $p(x)$ a polynomial of degree n, each of whose coefficients is 0 or ± 1, and which is divisible by $(x-1)^k$. Let q be a prime such that

$$\frac{q}{\ln q} < \frac{k}{\ln(n+1)}.$$

Prove that the complex qth roots of unity are roots of the polynomial $p(x)$.

36. Suppose f is a monic polynomial of degree n with complex coefficients, which has at least one root in common with each of its derivatives $f', f'', f^{(n-1)}$. Must f be a power of a linear polynomial?