PIGEONHOLE PROBLEMS

Try to solve these problems using a pigeonhole argument. It is not necessary to invoke the pigeonhole principle explicitly, but try to make its use apparent. Some of these problems can also be solved without pigeonhole reasoning. It's OK to hand in such a solution, but a pigeonhole argument is preferred.

1. Let A be any set of 19 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

2. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?

3. Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.) [To test your understanding, how many lattice points does one need in *four* dimensions to reach the same conclusion?]

4. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint nonempty subsets whose members have the same sum.

5. Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

6. Given any $n + 2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.

7. Given any $n+1$ distinct integers between 1 and $2n$, show that two of them are relatively prime. Is this result best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

8. Let G be a graph with n vertices and q edges. Assign the edges distinct labels $1, \ldots, q$ in some fashion. Then there exists a path (with repeated vertices allowed) of length at least $2q/n$ whose labels occur in increasing order.

9. There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

10. Given any $2n - 1$ integers, show that there are n of them whose sum is divisible by n. (Though superficially similar to some other pigeonhole problems, this problem is much more difficult and does not really involve the pigeonhole principle.)

11. Is it possible to cut an 8×8 chessboard with 13 straight lines (none passing through the midpoint of a square) such that every piece contains at most one midpoint of a square?

12. Let each of nine lines cut a square into two quadrilaterals whose areas are in the proportion $2 : 3$. Prove that at least three of the lines pass through the same point.

13. Let $a_1 < \cdots < a_n$, $b_1 > \cdots > b_n$, and $\{a_1, \ldots, a_n, b_1, \ldots, b_n\} = \{1, 2, \ldots, 2n\}$. Show that
$$\sum_{i=1}^{n} |a_i - b_i| = n^2.$$

14. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e., for any real number x and any $\epsilon > 0$, there is a element $y \in S$ such that $|x - y| < \epsilon$. (HINT. First let $x = 0$.)

15. Two disks, one smaller than the other, are each divided into 200 congruent sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100.

16. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

17. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

    [A *partition* of a set S is a collection of (nonempty) disjoint subsets (parts) whose union is S.]

18. (a) Let p be a prime such that there exists an integer h for which $h^2 + 1$ is divisible by p. (An odd prime has this property if and only if it is congruent to 1 modulo 4, but that's another story.) Prove that there exist integers a and b such that $p = a^2 + b^2$.

    HINT. Consider the numbers $u + vh$, where $0 \le u \le \lfloor \sqrt{p} \rfloor$ and $0 \le v \le \lfloor \sqrt{p} \rfloor$.

    NOTE. This is a minor variation of a standard application of the pigeonhole principle going back to Fermat. Do not hand in this problem if you've seen it or something similar before.

(b) Let p be a prime of the form $8k+1$. Prove that there exist positive integers a, b, m with $m < 2p$ such that $mp = a^4 + b^4$.

HINT. Show that there is an integer x such that $x^4 + 1$ is divisible by p, and consider the numbers $u + vx$, where $0 \leq u \leq \lfloor \sqrt{p} \rfloor$ and $0 \leq v \leq \lfloor \sqrt{p} \rfloor$.

(c) Improve the bound $m < 2p$. In particular, find a constant $c < 2$ such that one can take $m < cp$ for p large. (The best possible value of c requires some sophisticated number theory not involving the pigeonhole principle.)

19. Let $\mathbb{N}$ be the set of nonnegative integers. For any subset S of $\mathbb{N}$, let $P(S)$ be the set of all two-element subsets of S. Partition $P(\mathbb{N})$, arbitrarily, into two sets (of pairs) P_1 and P_2. Prove that $\mathbb{N}$ must contain an infinite subset S such that either $P(S)$ is contained in P_1 or $P(S)$ is contained in P_2.