Inequalities

Problem 1. For $p > 1$ and $a_1, a_2, \ldots, a_n$ positive, show that

$$\sum_{k=1}^{n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{n} a_k^p.$$

Problem 2. If $a_n > 0$ for $n = 1, 2, \ldots$, show that

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n,$$

provided that $\sum_{n=1}^{\infty} a_n$ converges.

Problem 3. For $n = 1, 2, 3, \ldots$ let

$$x_n = \frac{1000^n}{n!}.$$

Find the largest term of the sequence.

Problem 4. Suppose that $a_1, a_2, \ldots, a_n$ with $n \geq 2$ are real numbers greater than -1, and all the numbers a_j have the same sign. Show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) > 1 + a_1 + a_2 + \cdots + a_n.$$

Problem 5. If $a_1, \ldots, a_{n+1}$ are positive real numbers with $a_1 = a_{n+1}$, show that

$$\sum_{i=1}^{n} \left(\frac{a_i}{a_{i+1}} \right)^n \geq \sum_{i=1}^{n} \frac{a_{i+1}}{a_i}.$$

Problem 6. Show that for any real numbers $a_1, a_2, \ldots, a_n$,

$$\left(\sum_{i=1}^{n} \frac{a_i}{i} \right)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j - 1}.$$

Problem 7. Let $y = f(x)$ be a continuous, strictly increasing function of x for $x \geq 0$, with $f(0) = 0$, and let f^{-1} denote the inverse function to f. If a and b are nonnegative constants, then show that

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

Problem 8. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\min_{i<j}(a_i - a_j)^2 \leq M^2 \left(a_1^2 + \cdots + a_n^2\right),$$

where

$$M^2 = \frac{12}{n(n^2-1)}.$$

Problem 9. Let f be a continuous function on the interval $[0,1]$ such that $0 < m \leq f(x) \leq M$ for all x in $[0,1]$. Show that

$$\left(\int_0^1 \frac{dx}{f(x)}\right)\left(\int_0^1 f(x)dx\right) \leq \frac{(m+M)^2}{4mM}.$$

Problem 10. Consider any sequence $a_1, a_2, \ldots$ of real numbers. Show that

$$\sum_{n=1}^{\infty} a_n \leq \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{r_n}{n}\right)^{1/2}$$

where

$$r_n = \sum_{k=n}^{\infty} a_k^2.$$

(If the left-hand side of the inequality is ∞, then so is the right-hand side.)

Problem 11. Show that

$$\frac{1}{(n-1)!}\int_n^{\infty} w(t)e^{-t}dt < \frac{1}{(e-1)^n},$$

where t is real, n is a positive integer, and

$$w(t) = (t-1)(t-2)\cdots(t-n+1).$$

Problem 12. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

- There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge.
- $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$

Problem 13. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$ for all positive integers n.

Problem 14. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1, f'(0) = 0$ and for all $x \in [0, \infty)$, it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0$$

Prove that, for all $x \in [0, \infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

Problem 15. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0, 1]$.
(a) Show that $\int_0^1 f(x)dx \leq \frac{\pi}{4}$.
(b) Find such a function for which equality occurs.

Problem 16. For what pairs of positive real numbers (a, b) does the improper integral shown converge?

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) \mathrm{d}x$$

Problem 17. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \cdots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem 18. Let $f(x)$ be a continuous real-valued function de?ned on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|dx\, dy \geq \int_0^1 |f(x)|dx$$

Problem 19. For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(f) = \int_0^1 x\, (f(x))^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

Problem 20. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left|\int_0^{\alpha} f(x)\,dx\right| \le \frac{1}{8} \max_{0\le x\le 1} |f'(x)|$$

Problem 21. For $m \ge 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \le i < j < k \le m)$ is said to be area definite for $\mathbb{R}^n$ if the inequality

$$\sum_{1\le i<j<k\le m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \ge 0$$

holds for every choice of m points $A_1, \dots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

Problem 22. Let $X_1, X_2, ..$ be independent random variables with the same distribution, and let $S_n = X_1 + X_2 + ... + X_n, n = 1, 2, ...$ For what real numbers c is the following statement true:

$$\mathbb{P}\left(\left|\frac{S_{2n}}{2n} - c\right| \le \left|\frac{S_n}{n} - c\right|\right) \ge \frac{1}{2}.$$

Problem 23. Let a_n be a strictly increasing sequence of real number such that $a_n \le n^2 \ln(n)$ for all n. Prove that the series

$$\sum_{i=1}^{n} \frac{1}{a_{i+1} - a_i}$$

diverges.

Problem 24. Let $H_k = \sum_{i=1}^{k} \frac{1}{i}$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n} H_k}$$

has no real zeros.

Problem 25. Let f be a continuous, nonnegative function on $[0,1]$. Show that

$$\int_0^1 f(x)^3 dx \ge 4\left(\int_0^1 x f(x)^2 dx\right)\left(\int_0^1 x^2 f(x) dx\right)$$