ANALYSIS PROBLEMS

We use the notation $f(x) \sim g(x)$ to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. One says that $f(x)$ is *asymptotic* to $g(x)$.

Exercises.
These are problems whose techniques are worth knowing, but will generally only form components of larger solutions. (Submit at most one.)

1. Show that $\int_0^\infty \frac{\cos(ax)}{1+x^2} dx$ exists for $a \in \mathbb{R}$ and compute its value.
2. Find a simple function $f(x)$ for which $x^{1/x} - 1 \sim f(x)$ as $x \to \infty$.
3. For what pairs (a, b) of positive real numbers does the improper integral
$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$
converge?
4. Let a_n be the unique positive root of $x^n + x = 1$. Find a simple function $f(n)$ for which $1 - a_n \sim f(n)$ as $n \to \infty$.
5. For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)dx$ and $J(f) = \int_0^1 x f(x)^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

Problems.

1. For a positive real number a, calculate $\int_0^\infty t^{-1/2} e^{-a(t+t^{-1})} dt$.
2. Let f be a function on $[0, \infty)$, differentiable and satisfying
$$f'(x) = -3f(x) + 6f(2x)$$
for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to ∞ as x increases). For n a nonnegative integer, define
$$\mu_n = \int_0^\infty x^n f(x) dx$$
(the nth moment of f).
   (a) Express μ_n in terms of μ_0.
   (b) Prove that the sequence $\{\mu_n \cdot 3^n/n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.
3. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,
$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$
If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.
4. Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that
$$a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$
for all u with $0 < |u| \leq c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

5. The function $K(x,y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x)$$

and

$$\int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

6. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx.$$

7. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \ge 0$ for all real x. Prove that $|f(x)|$ is bounded.

8. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \le C \int_{-1}^1 |p(x)|\,dx.$$

9. Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L.$$

10. Let $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1,b_1)x^{a_1}y^{b_1} + (a_2,b_2)x^{a_2}y^{b_2} + \cdots + (a_n,b_n)x^{a_n}y^{b_n} = (0,0).$$

11. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

12. Let f be a real-valued function having partial derivatives and which is defined for $x^2 + y^2 \le 1$ and is such that $|f(x,y)| \le 1$. Show that there exists a point (x_0,y_0) in the interior of the unit circle for which

$$\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2 \le 16.$$

13. (a) On $[0,1]$, let f have a continuous derivative satisfying $0 < f'(x) \le 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x)dx\right)^2 \ge \int_0^1 f(x)^3 dx.$$

(b) Find an example where equality occurs.

14. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t)\sin t dt = \int_0^x P(t)\cos t dt$$

has only finitely many real solutions x.

15. Let C be the class of all real valued continuously differentiable functions f on the interval $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \leq \int_0^1 |f'(x) - f(x)| dx$$

for all $f \in C$.

16. Given a convergent series $\sum a_n$ of positive terms, prove that the series $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ must also be convergent.

17. Given that $f(x) + f'(x) \to 0$ as $x \to \infty$, prove that both $f(x) \to 0$ and $f'(x) \to 0$.

18. Suppose that $f''(x)$ is continuous on $\mathbb{R}$, and that $|f(x)| \leq a$ on $\mathbb{R}$, and $|f''(x)| \leq b$ on $\mathbb{R}$. Find the best possible bound $|f'(x)| \leq c$ on $\mathbb{R}$.

19. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x. (Note that we cannot replace 2 by 1 because of the function $f(x) = e^x$.)

20. Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2)\, dx$$

converges.

21. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

22. Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

23. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

24. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

25. Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

26. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

27. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

28. Let $f : [0, 1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y)\,dy$, $b = \int_0^1 f(1, y)\,dy$, $c = \int_0^1 f(x, 0)\,dx$, $d = \int_0^1 f(x, 1)\,dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

29. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

30. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x + n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

31. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \le M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

32. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

33. Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?