18.A34 PROBLEMS #3

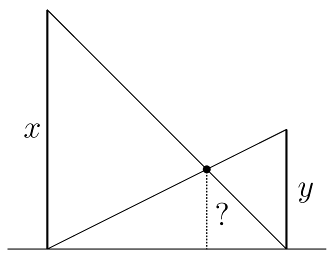
28. [1] Let $x, y > 0$. The *harmonic mean* of x and y is defined to be $2xy/(x+y)$. The *geometric mean* is $\sqrt{xy}$. The *arithmetic mean* (or *average*) is $(x+y)/2$. Show that

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2},$$

with equality if and only if $x = y$.

29. [1] A car travels one mile at a speed of x mi/hr and another mile at y mi/hr. What is the average speed? What kind of mean of x and y is this?

30. [1] Consider two telephone poles of heights x and y. Connect the top of each pole to the bottom of the other with a rope. What is the height of the point where the ropes cross? What kind of mean is this related to?



31. [1] Given two line segments of lengths x and y, describe a simple geometric construction for constructing a segment of length $\sqrt{xy}$.

32. [1] Suppose x and y are real numbers such that $x^2 + y^2 = x + y$. What is the largest possible value of x?

33. [2.5] (a) Let $x, y > 0$ and $p \neq 0$. The *p-th power mean* of x and y is defined to be

$$M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}.$$

Note that $M_{-1}(x,y)$ is the harmonic mean and $M_1(x,y)$ is the arithmetic mean. If $p < q$, then show that

$$M_p(x,y) \leq M_q(x,y),$$

with equality if and only if $x = y$.

(b) Compute $\lim_{p\to\infty} M_p(x,y)$, $\lim_{p\to 0} M_p(x,y)$, $\lim_{p\to -\infty} M_p(x,y)$.

34. [3.5] Let $x, y > 0$. Define two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ as follows:

$$\begin{aligned} x_1 &= x & y_1 &= y \\ x_{n+1} &= \frac{x_n + y_n}{2} & y_{n+1} &= \sqrt{x_n y_n}, \quad n > 1. \end{aligned}$$

It's not hard to see that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. This limit is denoted $AG(x,y)$ and is called the *arithmetic-geometric mean* of x and y. Show that

$$AG(x,y) = \frac{\pi}{\displaystyle\int_0^{\pi} \frac{d\theta}{\sqrt{x^2\sin^2\theta + y^2\cos^2\theta}}}.$$

35. Let $x > 0$. Define

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}.$$

More precisely, let $x_1 = x$ and $x_{n+1} = x^{x_n}$ if $n > 1$, and define $f(x) = \lim_{n\to\infty} x_n$.

(a) [1] Compute $f(\sqrt{2})$.

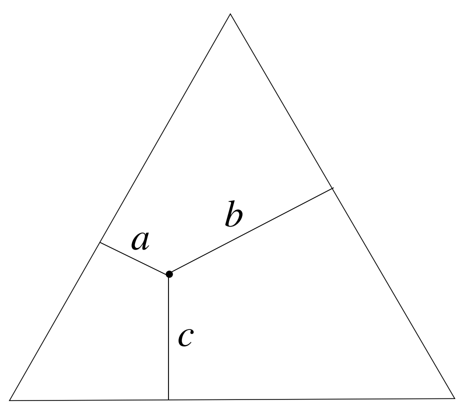
(b) [3.5] For what values of x does $f(x)$ exist?

(c) [3.5] Let

$$\begin{aligned} f(x+1) &= \sum_{n\geq 0} a_n \frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 9\frac{x^3}{3!} + 56\frac{x^4}{4!} + 480\frac{x^5}{5!} + 5094\frac{x^6}{6!} + \cdots. \end{aligned}$$

Show that a_n is a positive integer.

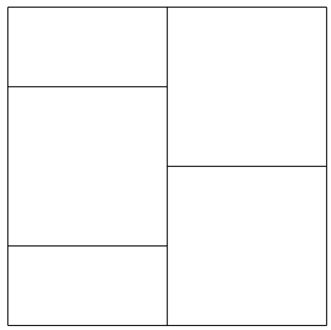
36. [2] Let T be an equilateral triangle. Find all points x in T that minimize the sum $a+b+c$ of the distances a, b, c of x from the three sides of T.



37. [2.5] Alice and Bob play the following game. They begin with a sequence $(a_1, \ldots, a_{2n})$ of positive integers. The players alternate turns, with Alice moving first. When it is someone's turn to move, that person can remove either the first or last remaining term of the sequence. A player's score at the end is the sum of his/her chosen numbers. Show that Alice has a strategy that guarantees a score at least as large as Bob's.

38. [3] Let F_n denote the nth Fibonacci number. Let p be a prime not equal to 5. Show that either F_{p-1} or F_{p+1} is divisible by p. Which?

39. Fix an integer $n > 0$. Let $f(n)$ be the most number of rectangles into which a square can be divided so that every line which is parallel to one of the sides of the square intersects the interiors of at most n of the rectangles. For instance, in the following figure there are five rectangles, and every horizontal or vertical line intersects the interior of at most 3 of them. This is not best possible, since we can obviously do the same with nine rectangles.



(a) [2] It is obvious that $f(n) \geq n^2$. Show that in fact $f(n) > n^2$ for $n \geq 3$.

(b) [3] Show that $f(n) < \infty$, i.e., for any fixed n we cannot divide a square into *arbitrarily many* rectangles with the desired property. In fact, one can show $f(n) \leq n^n$.

(c) [5] Find the actual value of $f(n)$. The best lower bound known is $f(n) \geq 3 \cdot 2^{n-1} - 2$.