

ANALYSIS PROBLEMS

We use the notation $f(x) \sim g(x)$ to mean $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$. One says that $f(x)$ is *asymptotic* to $g(x)$.

1. For a positive real number a , calculate $\int_0^\infty t^{-1/2} e^{-a(t+t^{-1})} dt$.
2. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(the n th moment of f).

- (a) Express μ_n in terms of μ_0 .
 - (b) Prove that the sequence $\{\mu_n \cdot 3^n / n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.
3. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y ,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x .

4. Let a and b be positive numbers. Find the largest number c , in terms of a and b , such that

$$a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \leq c$ and for all x , $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

5. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x , $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y) dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y) dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

6. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

7. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

8. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x . Prove that $|f(x)|$ is bounded.

9. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)| dx.$$

10. Let $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

11. Show that all solutions of the differential equation $y'' + e^xy = 0$ remain bounded as $x \rightarrow \infty$.
12. Let f be a real-valued function having partial derivatives and which is defined for $x^2 + y^2 \leq 1$ and is such that $|f(x, y)| \leq 1$. Show that there exists a point (x_0, y_0) in the interior of the unit circle for which

$$\left(\frac{\partial f}{\partial x}(x_0, y_0) \right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0) \right)^2 \leq 16.$$

13. (a) On $[0, 1]$, let f have a continuous derivative satisfying $0 < f'(x) \leq 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left(\int_0^1 f(x) dx \right)^2 \geq \int_0^1 f(x)^3 dx.$$

- (b) Find an example where equality occurs.

14. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t) \sin t dt = \int_0^x P(t) \cos t dt$$

has only finitely many real solutions x .

15. Let C be the class of all real valued continuously differentiable functions f on the interval $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$. Determine the largest real number u such that

$$u \leq \int_0^1 |f'(x) - f(x)| dx$$

for all $f \in C$.

16. Given a convergent series $\sum a_n$ of positive terms, prove that the series $\sum \sqrt[n]{a_1 a_2 \cdots a_n}$ must also be convergent.
17. Maximize $2^{-x} + 2^{-1/x}$ over $(0, \infty)$.
18. Find a simple function $f(x)$ for which $x^{1/x} - 1 \sim f(x)$ as $x \rightarrow \infty$.
19. Let a_n be the unique positive root of $x^n + x = 1$. Find a simple function $f(n)$ for which $1 - a_n \sim f(n)$ as $n \rightarrow \infty$.
20. Given that $f(x) + f'(x) \rightarrow 0$ as $x \rightarrow \infty$, prove that both $f(x) \rightarrow 0$ and $f'(x) \rightarrow 0$.
21. Suppose that $f''(x)$ is continuous on \mathbb{R} , and that $|f(x)| \leq a$ on \mathbb{R} , and $|f''(x)| \leq b$ on \mathbb{R} . Find the best possible bound $|f'(x)| \leq c$ on \mathbb{R} .
22. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x . Suppose that $f'''(x) \leq f(x)$ for all x . Show that $f'(x) < 2f(x)$ for all x . (Note that we cannot replace 2 by 1 because of the function $f(x) = e^x$.)
23. Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

24. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base- b digits of n . For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

25. Evaluate

$$\lim_{x \rightarrow 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n} \right)^{x^n}.$$

26. Find all differentiable functions $f : (0, \infty) \rightarrow (0, \infty)$ for which there is a positive real number a such that

$$f' \left(\frac{a}{x} \right) = \frac{x}{f(x)}$$

for all $x > 0$.

27. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{a_n^{k+1}}{n^k}.$$

28. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

29. Find all continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every rational number q , the number $f(q)$ is rational and has the same denominator as q . (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: \gcd means greatest common divisor.)

30. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, & f(0) &= 1, \\ g' &= fg^2h + \frac{4}{fh}, & g(0) &= 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, & h(0) &= 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

31. Let $f : [0, 1]^2 \rightarrow \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y) dy$, $b = \int_0^1 f(1, y) dy$, $c = \int_0^1 f(x, 0) dx$, $d = \int_0^1 f(x, 1) dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

32. Let $f : (1, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)} \quad \text{for all } x > 1.$$

Prove that $\lim_{x \rightarrow \infty} f(x) = \infty$.

33. Find all differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n .

34. Suppose that the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)$$

for some constants a, b . Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

35. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x \rightarrow \infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} dx$ diverges.

36. Is there a strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x ?