Problem Set 6. Due 10/23

Reminder: You must acknowledge your sources and collaborators (even if it is "none", you must write so). Failure to do so on this problem set will result in an automatic 2-point deduction.

1. Using a computer algebra system (e.g., WOLFRAM ALPHA[1], MATHEMATICA, SAGEMATH, MAPLE), compute the Bell number B_{10} via its exponential generating function e^{e^x-1}. Include a screenshot/printout of your code and output.

   This method allows you to compute a sequence from its generating function. You should check (but no need to submit) that your methods correctly computes the first few values of the Bell numbers: $(B_0, B_1, B_2, \dots) = (1, 1, 2, 5, 15, \dots)$. You are also encouraged to use this method to check your answers to the remaining problems.

2. Let $g_0 = g_1 = 1$ and $g_n = ng_{n-1} + n(n-1)g_{n-2}$ for $n \geq 2$. Find the exponential generating function $\sum_{n\geq 0} g_n x^n/n!$.

3. Let $g_0 = g_1 = 1$ and
$$g_{n+1} = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} g_k g_{n-k}, \quad \text{for all } n \geq 1.$$
   Express the exponential generating function $\sum_{n\geq 0} g_n x^n/n!$ in closed form. Your answer should involve some trigonometric functions.

4. Let $c(n, k)$ be the number of permutations of $[n]$ with exactly k cycles (recall that this is the signless Stirling numbers of the first kind). For fixed $k \geq 1$, determine the exponential generating function $\sum_{n\geq 0} c(n, k)x^n/n!$.

5. Let g_n be the number of ways that a group of n children can form into circles by holding hands, and one child stands in the center of each circle. A circle may consists of as few as one child clasping his or her hands), but each circle must contain a child inside it. The first few terms are $(g_0, g_1, g_2, \dots) = (1, 0, 2, 3, 20, 90, \dots)$. Determine the exponential generating function $\sum_{n\geq 0} g_n x^n/n!$.

6. Let g_n denote the number of ways to partition $[n]$ and then partition each block. Equivalently, g_n is the number of pairs (Π_1, Π_2) of partitions of $[n]$ where every block in Π_1 is a union of blocks of Π_2 (an example: $\Pi_1 = \{\{1, 3, 4\}, \{2, 5, 8\}, \{6, 7\}\}$ and $\Pi_2 = \{\{1\}, \{2, 5, 8\}, \{3, 4\}, \{6\}, \{7\}\}$). The first few terms are $(g_0, g_1, g_2, \dots) = (1, 1, 3, 12, 60, \dots)$. Determine the exponential generating function $\sum_{n\geq 0} g_n x^n/n!$.

7. Let g_n denote the number of 2-regular graphs on n labeled vertices. The first few terms are $(g_0, g_1, g_2, \dots) = (1, 0, 0, 1, 3, 12, 70, \dots)$. Determine the exponential generating function $\sum_{n\geq 0} g_n x^n/n!$.

8. Let g_n denote the number of rooted trees on n labeled vertices.[2] The first few terms are $(g_0, g_1, g_2, g_3, \dots) = (0, 1, 2, 9, 64, \dots)$. Let $G(x) = \sum_{n\geq 0} g_n x^n/n!$. Prove that $G(x) = xe^{G(x)}$.

[1] In WOLFRAM ALPHA, type `power series` followed by the generating function. Click on more terms.

[2] *Rooted* means that one of the vertices is marked as the "root". Two rooted labeled trees are considered identical if they are isomorphic as labeled graphs and furthermore their roots are at the same labeled vertex.