Problem Set 3. Due 9/25

1. What is the number of permutations of $[n]$ in which the numbers 1 and 2 are part of the same 3-cycle?

2. What is the number of permutations of $[2n]$ whose longest cycle has length exactly n?

3. What is the number of permutations of $[6]$ whose cube is the identity?

4. This problem is a variation of a result discussed in class.[1] We have the same 100 prisoners and evil warden. As before, the prisoners are brought one at a time into a room with 100 boxes labeled $1, 2, \ldots, 100$. Inside each box is the name of a prisoner, a different name in each box. This time a prisoner must open 99 boxes, one at a time. If the prisoner encounters his own name, then all prisoners are killed. The prisoners may talk together before the first prisoner enters the room with the boxes. After that there is no further communication. A prisoner cannot leave a signal in the room. All boxes are closed before each prisoner enters the room. Clearly the probability that the prisoners aren't killed cannot exceed $1/100$, since that is the probability that the first prisoner does not encounter his name. What strategy maximizes the probability that the prisoners are not killed, and what is this maximum probability?

5. Let $a(n, k)$ be the number of permutations of $[n]$ with k cycles in which the entries 1 and 2 are in the same cycle. Prove that for $n \geq 2$,

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)\cdots(x+n-1).$$

6. Prove that the number of permutations of $[n+1]$ with exactly two cycles is

$$n!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

7. Let $n \geq 3$. Pick a permutation π of $[n]$ uniformly at random (i.e., all permutations are equally likely). What is the probability that $1, 2, 3$ are all in different cycles of π?

[1] For the class problem, see `https://www.math.dartmouth.edu/~pw/solutions.pdf`, Problem 1.