Practice Midterm 2

Time: 80 minutes.
6 problems worth 10 points each.
No electronic devices. You may bring one sheet of notes on letter-sized paper (front and back) **in your own handwriting**. Typed, printed, or photocopied notes are **forbidden**.

You must provide justification in your solutions (not just answers). You may quote theorems and facts proved in class, course textbook/notes, or homework, provided that you state the facts that you are using.

1. There are n soldiers standing in a line. We wish to do all of the following:

   - Cut line in a number of places to divide the soldiers into at least two groups;
   - Select a commander within each group;
   - Select a captain among the commanders.

   Let g_n be the number of ways to do this. Determine the generating function for g_n (you may choose to give either the ordinary generating function or the exponential generating function. You do not need to solve for g_n. It is sufficient to write down a correct closed form expression for the generating function; you do not need to simplify for this problem).

   Solution. We solvefor the ordinary generating function $G(x) = \sum_{n \geq 0} g_n x^n$. By the compositional formula, one has $G(x) = B(A(x))$, where $A(x)$ is the generating function for the sequence

   $$a_n = n \quad \text{for all } n \geq 0,$$

   since this is the number of ways to select a commander in an n-person group, and $B(x)$ is the generating function for the sequence

   $$b_n = \begin{cases} n & \text{if } n \geq 2 \\ 0 & \text{if } n = 0, 1 \end{cases},$$

   as this is the number of ways to select a captain when where are n groups with pre-chosen commanders (we set $b_n = 0$ to forbid having fewer than zero groups).

   We have

   $$A(x) = \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} n x^n = \frac{x}{(1-x)^2}$$

   (recall that we derived this formula in class by differentiating $\frac{1}{1-x} = 1 + x + x^2 + \cdots$) and

   $$B(x) = \sum_{n \geq 0} b_n x^n = \sum_{n \geq 2} n x^n = \frac{x}{(1-x)^2} - x.$$

   So the desired generating function is

   $$\boxed{G(x) = \sum_{n \geq 0} g_n x^n = B(A(x)) = \frac{\frac{x}{(1-x)^2}}{\left(1 - \frac{x}{(1-x)^2}\right)^2} - \frac{x}{(1-x)^2} = \frac{x(1-x)^2}{(1-3x+x^2)^2} - \frac{x}{(1-x)^2}.}$$

2. Let g_n denote the number of label graphs on vertex set $[n]$ with maximum degree at most 2, at least two connected components, and no isolated vertices. Determine $\sum_{n\geq 0} g_n x^n/n!$.

Solution. Let $G(x) \sum_{n\geq 0} g_n x^n/n!$. Note that having maximum degree at most 2 is equivalent to having all connected components be paths and cycles (why?). Applying the compositional formula for exponential generating functions, we have $G(x) = B(A(x))$, where A is the exponential generating function for the sequence a_n, with a_n being the number of labeled paths and cycles on n labeled vertices, forbidding the possibility of an isolated vertex.

Note that there are $(n-1)!/2$ ways to form a cycle for all $n \geq 3$ (we need at least 3 vertices to form a cycle, and note that the orientation of the cycle is not considered, hence dividing by 2). Likewise, there are $n!/2$ ways to form a path on $n \geq 2$ labeled vertices. Thus

$$a_n = \begin{cases} 0 & \text{if } n = 0, 1, \\ 1 & \text{if } n = 2 \\ \frac{(n-1)!}{2} + \frac{n!}{2} & \text{if } n \geq 3. \end{cases}$$

Thus (here we use the familiar series $-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}$)

$$\begin{aligned} A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!} &= \frac{x^2}{2} + \sum_{n\geq 3} \frac{x^n}{2n} + \sum_{n\geq 3} \frac{x^n}{2} \\ &= \frac{x^2}{2} + \frac{1}{2}\left(-\log(1-x) - x - \frac{x^2}{2}\right) + \frac{x^3}{2(1-x)} \\ &= -\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{2(1-x)} - \frac{1}{2}\log(1-x). \end{aligned}$$

On the other hand, since we require at least two connected components, $B(x)$ is the exponential generating function for the sequence b_n where $b_0 = b_1 = 0$ and $b_n = 1$ for all $n \geq 2$. So

$$B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 2} \frac{x^n}{n!} = e^x - 1 - x.$$

Thus

$$\begin{aligned} G(x) = B(A(x)) &= \exp\left(-\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{2(1-x)} - \frac{1}{2}\log(1-x)\right) - 1 + \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{2(1-x)} + \frac{1}{2}\log(1-x) \\ &= \boxed{\frac{\exp\left(-\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{2(1-x)}\right)}{\sqrt{1-x}} - 1 + \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{2(1-x)} + \frac{1}{2}\log(1-x)} \end{aligned}$$

3. (a) Let $p_{\leq k}(n)$ denote the number of partitions of n with at most k parts. Determine the generating function

$$P_{\leq k}(x) = \sum_{n\geq 0} p_{\leq k}(n) x^n.$$

(Your answer may contain at most one summation or product.)

Solution. This was done in lecture. By conjugating, we see that $p_{\leq k}(n)$ also equals to the number of partitions of n with all parts at most k, and thus

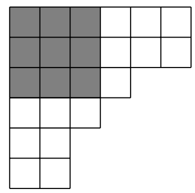
$$P_{\leq k}(x) = \sum_{n\geq 0} p_{\leq k}(n) x^n = \boxed{\prod_{j=1}^{k} \frac{1}{1-x^k}}.$$

(b) Let $q(n)$ denote the number of self-conjugate partitions. Prove that

$$\sum_{n\geq 0} q(n)x^n = \sum_{k\geq 0} x^{k^2} P_{\leq k}(x^2).$$

(Recall that a partition is *self-conjugate* if its Ferrers shape is is mirror-symmetric along its main diagonal.)

Solution. Consider the largest top-left aligned square contained in the Ferrers shape of a partition (this is called the *Durfee square*). E.g., for the partition $(6,6,4,3,2,2)$, the largest such square has width 3.



Note that by removing the Durfee square, calling its width k, we obtain (to its right) a partition λ with at most k parts, and also (below the Durfee square) the conjugate of λ. This is gives a bijection between self-conjugate partitions and pairs (k,λ), where k nonnegative integer, and λ is a partition with at most k parts (consider the partition to the right of the Durfee square). Thus the generating function for the number of self-conjugate partitions whose Durfee square has width k is

$$x^{k^2} \sum_{n\geq 0} p_{\leq k}(n) x^{2n} = x^{k^2} P_{\leq k}(x^2).$$

Summing over all nonnegative integers k yields the claimed result.

Remark 1. You should check that a modification of this argument also shows the identity

$$\sum_{n\geq 0} p(n)x^n = \sum_{k\geq 0} x^{k^2} P_{\leq k}(x)^2.$$

Remark 2. We showed in lecture that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts. Thus

$$\sum_{n\geq 0} q(n)x^n = \prod_{k\geq 1}(1+x^{2k-1}).$$

4. Let T_1 and T_2 be two distinct spanning trees of G with $T_1 \neq T_2$. Prove that there exist edges $e \in E(T_1)\setminus E(T_2)$ and $f \in E(T_2)\setminus E(T_1)$ so that $T_1 - e + f$ and $T_2 - f + e$ are both spanning trees in G.

(Here $T_i - e + f$ is the subgraph obtained from T_i by removing the edge e and adding the edge f.)

Solution. Pick an arbitrary edge $e = xy \in E(T_1)\setminus E(T_2)$ (such an edge must exist since neither T_1 is not contained in T_2). Removing e from T_1 disconnects T_1 into exactly two components, which we call C_x and C_y, where $x \in C_x$ and $y \in C_y$. Consider the unique path P in T_2 from x to y. Since the path P starts in C_x and ends in C_y, P contains an edge f with one endpoint in C_x and the other in C_y. In particular, $f \in E(P) \subset E(T_2)$. Also, $f \notin E(T_1)$, since otherwise removing e from T_1 would not have disconnected C_x from C_y. So $f \in E(T_2)\setminus E(T_1)$.

We see that $T_1 - e + f$ is a spanning tree since adding f to $T_1 - e$ joins its two connected components C_x and C_y.

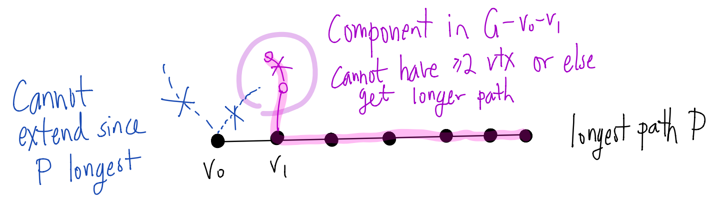
Also, $T_2 - f + e$ is a spanning tree since $T_2 + e$ contains the cycle $P + e$, and so it remains connected after removing f from the cycle.

(In both cases we are using that a connected graph with n vertices and $n-1$ edges is a tree.)

5. Let G be a connected graph with at least 3 vertices. Prove that there exist two distinct vertices x, y in G such that $G - x - y$ is connected and the distance between x and y is at most 2.

   (Recall that the *distance* between a pair vertices is the length of the shortest path between the two vertices, where the *length* of a path is the number of edges on the path. Here $G - x$ is the graph obtained from G by removing the vertex x along with all edges incident to x.)

   Solution. Let $P = v_0 v_1 \cdots v_k$ be a path of maximum length in G (always a good thing to try!). If $G - v_0 - v_1$ is connected, then choosing $x = v_0$ and $y = v_1$ works. So let us assume that $G - v_0 - v_1$ is not connected. Since P is a longest path, it cannot be extended from v_0, and so all neighbors of v_0 in G are contained in P. Since $G - v_0 - v_1$ is not connected, it has some component C other than the one containing $P - v_0 - v_1$. Then C has a vertex adjacent to v_1 in G. If C has more than one vertex, then one could find a path in G longer than P by rerouting P into C via v_1. Thus C has only one vertex, and let y be this vertex and $x = v_0$. Then x and y have distance at most 2 (via v_1), and their removal does not disconnect G.



6. Let $k \geq 2$. Prove that every k-regular connected bipartite graph is 2-connected.

   Solution. For contradiction, let G be a k-regular connected bipartite graph that is not 2-connected. Thus G has a cut-vertex v. Let us label the bipartition of the vertex set of G by $A \cup B$, so that all edges of G have one vertex in A and the other vertex in B. We may assume, without loss of generality, that $v \in A$. Since v is a cut vertex, its removal disconnects the remaining vertices into two components. Let $A = A_1 \cup A_2 \cup \{v\}$ and $B = B_1 \cup B_2$, where $A_1 \cup B_1$ form one component of $G - v$ and $A_2 \cup B_2$ induce the other component.

   All neighbors of v lie in B. Suppose that k_1 neighbors of v lie in B_1, where $0 < k_1 < k$, and the remaining $k_2 = k - k_1$ neighbors lie in B_2. We can calculate the number of edges between A_1 and B_1 in two ways. By summing over degrees in A_1, we see that there are $|A_1|k$ edges between A_1 and B_1. On the other hand, by summing over degrees in B_1 and subtracting the k_1 edges between k_1 and B_1, we see that there are exactly $|B_1|k - k_1$ edges between A_1 and B_1. So $|A_1|k = |B_1|k - k_1$, which is impossible since the LHS is divisible by k while the RHS is not. This is a contradiction.