Practice Midterm 2

Time: 80 minutes.

6 problems worth 10 points each.

No electronic devices. You may bring one sheet of notes on letter-sized paper (front and back) **in your own handwriting**. Typed, printed, or photocopied notes are **forbidden**.

You must provide justification in your solutions (not just answers). You may quote theorems and facts proved in class, course textbook/notes, or homework, provided that you state the facts that you are using.

1. There are n soldiers standing in a line. We wish to do all of the following:

   - Cut line in a number of places to divide the soldiers into at least two groups;
   - Select a commander within each group;
   - Select a captain among the commanders.

   Let g_n be the number of ways to do this. Determine the generating function for g_n (you may choose to give either the ordinary generating function or the exponential generating function. You do not need to solve for g_n. It is sufficient to write down a correct closed form expression for the generating function; you do not need to simplify for this problem).

2. Let g_n denote the number of label graphs on vertex set $[n]$ with maximum degree at most 2, at least two connected components, and no isolated vertices. Determine $\sum_{n\geq 0} g_n x^n/n!$.

3. (a) Let $p_{\leq k}(n)$ denote the number of partitions of n with at most k parts. Determine the generating function

   $$P_{\leq k}(x) = \sum_{n\geq 0} p_{\leq k}(n) x^n.$$

   (Your answer may contain at most one summation or product.)

   (b) Let $q(n)$ denote the number of self-conjugate partitions. Prove that

   $$\sum_{n\geq 0} q(n)x^n = \sum_{k\geq 0} x^{k^2} P_{\leq k}(x^2).$$

   (Recall that a partition is *self-conjugate* if its Ferrers shape is is mirror-symmetric along its main diagonal.)

4. Let T_1 and T_2 be two distinct spanning trees of G with $T_1 \neq T_2$. Prove that there exist edges $e \in E(T_1) \setminus E(T_2)$ and $f \in E(T_2) \setminus E(T_1)$ so that $T_1 - e + f$ and $T_2 - f + e$ are both spanning trees in G.

   (Here $T_i - e + f$ is the subgraph obtained from T_i by removing the edge e and adding the edge f.)

5. Let G be a connected graph with at least 3 vertices. Prove that there exist two distinct vertices x, y in G such that $G - x - y$ is connected and the distance between x and y is at most 2.

   (Recall that the *distance* between a pair vertices is the length of the shortest path between the two vertices, where the *length* of a path is the number of edges on the path. Here $G - x$ is the graph obtained from G by removing the vertex x along with all edges incident to x.)

6. Let $k \geq 2$. Prove that every k-regular connected bipartite graph is 2-connected.