Practice Midterm 1

Closed book. No notes/calculators/phones.
Time: 80 minutes.
6 problems worth 10 points each.
You must provide justification in your solutions (not just answers). Simplify all answers and express in closed form whenever possible.

1. Let $n \geq 3$ be a positive integer. Determine the number of solutions to $x + y + z \leq n$ with integers $x, y, z \geq 1$.

2. Prove that for all positive integers n,
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

3. Let $D(n)$ denote the number of derangements (permutations without fixed points) of $[n]$. Give a combinatorial proof of the identity
$$D(n+1) = n(D(n) + D(n-1)), \quad \text{for all } n \geq 1.$$
Do not use the formula for the numbers $D(n)$ derived in class.

4. Let $n \geq 4$ be a positive integer. How many permutations of $[n]$ are there such that some cycle contains both 1 and 2 and a different cycle contains both 3 and 4?

5. Let $a_0 = 0$ and $a_{n+1} = 3a_n + n$ for all $n \geq 0$.
   (a) Express the generating function $A(x) = \sum_{n \geq 0} a_n x^n$ in closed form.
   (b) Find a closed form formula for a_n.

6. Let a_n be the number of partitions of n whose parts differ by at least two. For instance, when $n = 10$ the partitions are (10), $(9, 1)$, $(8, 2)$, $(7, 3)$, $(6, 4)$, $(6, 3, 1)$.

   Let b_n be the number of partitions of n whose smallest part is at least as large as the number of parts. For instance, when $n = 10$ the partitions are (10), $(8, 2)$, $(7, 3)$, $(6, 4)$, $(5, 5)$, $(4, 3, 3)$.

   Give a bijective proof that $a_n = b_n$.
   Hint. Consider $1 + 3 + 5 + \cdots + (2k - 1)$.